Eidgenössische Technische Hochschule Zürich
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Partitions and Packings of Complete Geometric Graphs with Plane Spanning Double Stars and Paths

Master Thesis

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Abstract

Consider the following problem: Given a complete geometric graph with an even number of vertices, can its edge set be partitioned into plane spanning trees?

In the main part of this thesis we investigate this question for *plane spanning double stars* instead of general spanning trees. We give a necessary, as well as a sufficient condition for the existence of a partition into plane spanning double stars. We also construct complete geometric graphs with an even number of vertices that cannot be partitioned into plane spanning double stars.

We then consider the more general problem of packing plane spanning double stars into complete geometric graphs. We show that finding a packing with plane spanning double stars is equivalent to finding an induced subgraph that can be partitioned into plane spanning double stars. We use this to find large packings with plane spanning double stars in several special point sets.

In the last part of the thesis, we investigate the above question for *plane spanning paths*. We consider complete geometric graphs with only one vertex not on the boundary of the convex hull, and we give for these graphs a necessary and sufficient condition for the existence of a partition into plane spanning paths.

Finally, we show a complexity result about the more general problem of finding colorings of line segment arrangements without monochromatic crossings.

Contents

Part I

Introduction and Survey

Chapter 1

Introduction

This thesis is motivated by the following question posed by Bose, Hurtado, Rivera-Campo and Wood [6]:

Question 1.1 *Does every complete geometric graph with an even number of vertices allow a partition of its edge set into plane spanning trees?*

Note that a complete geometric graph on n vertices has $\binom{n}{2} = \frac{n}{2}(n-1)$ edges, while each spanning tree has $n-1$ edges. Thus a partition of the edge set into spanning trees consists of $\frac{n}{2}$ pairwise edge-disjoint spanning trees, which is why we require the number of vertices to be even.

We will discuss this question for special kinds of trees, called double stars, as well as for paths. We will also consider the more general problem of partitioning the elements of a line segment arrangements into non-crossing sub-arrangements.

1.1 Definitions

A *point set* $\mathcal{P}$ is a finite set of points in $\mathbb{R}^2$. We say that a point set is in *general position* if no three points lie on a line. For the rest of the thesis, we will assume all point sets to be in general position. We denote the convex hull of $\mathcal{P}$ by $Conv(\mathcal{P})$. An *empty convex n-gon* in a point set $\mathcal{P}$ is a subset $\mathcal{P}'$ of n points in convex position such that no point of $\mathcal{P}$ lies in the interior of $Conv(\mathcal{P}')$.

Following Pilz and Welzl [17], for two point sets $\mathcal{P}$ and $\mathcal{P}'$ of equal size, we call a bijection φ from $\mathcal{P}$ to $\mathcal{P}'$ *crossing-preserving* if for every crossing pair of line segments (p,q) and (r,s), defined by points p,q,r and s in $\mathcal{P}$, the line segments $(\varphi(p),\varphi(q))$ and $(\varphi(r),\varphi(s))$ cross as well. If there exists such a bijection, we say that the point set $\mathcal{P}'$ *crossing-dominates* the point set $\mathcal{P}$. If $\mathcal{P}'$ induces more crossings than $\mathcal{P}$, then $\mathcal{P}'$ *strictly crossing-dominates* the point set $\mathcal{P}$. The point sets that are not strictly crossing-dominated by any other point set are called *crossing-maximal*.

A *geometric graph* is a drawing of a graph in $\mathbb{R}^2$ with straight-line edges, i.e. the vertex set is a point set and each edge is a line segment. A geometric graph is called *plane* or *crossing-free* if no pair of edges crosses. For two vertices v and w in a geometric graph G, we say that v *sees* w *in* G if the line segment between v and w is not crossed by any edge of G. Note that we neither require nor forbid that (v, w) is an edge of G.

The complete geometric graph $K(\mathcal{P})$ of a point set $\mathcal{P}$ is the geometric graph obtained by drawing a line segment between any two points in $\mathcal{P}$.

A *caterpillar* is a tree such that the induced subgraph of the vertex set without the leaves is a path. This induced subgraph is called the *spine*. A *double star* is either a single edge, or a caterpillar whose spine consists of a single edge. A caterpillar C is *symmetric* if it has an edge (v, w) such that there is a graph-isomorphism between the components A and B of $C \setminus (v, w)$, with $v \in A$ and $w \in B$, that maps v to w.

A *partition* of the edge set of a graph is a grouping of the edges into subgraphs in such a way that every edge is part of exactly one subgraph. A *packing* of the edge set of a graph is a grouping of the edges into subgraphs in such a way that every edge appears in at most one subgraph. A *covering* of the edge set of a graph is a grouping of the edges into subgraphs in such a way that every edge is part of at least one subgraph. In other words, a partition of the edge set of a graph is both a packing and a covering.

1.2 Structure

The thesis consists of three parts. In the first part we give an introduction to the topic and the necessary definitions. We also discuss some related results in Chapter 2.

The second part of the thesis is about double stars. Chapter 3 discusses partitions of complete geometric graphs into plane spanning double stars. We show that the spines of the double stars form a perfect matching with certain properties. We also show that, given a perfect matching in a complete geometric graph, one can check in polynomial time whether the edges in this perfect matching are the spines of a partition of the complete geometric graph into plane spanning double stars. Finally, we give an example of a point set whose complete geometric graph cannot be partitioned into plane spanning double stars.

In Chapter 4 we consider packings of plane spanning double stars. We show that finding a packing with k spanning double stars is equivalent to finding a matching with k edges that satisfies the same conditions as the perfect matching for partitions. We prove a lower bound for the expected number of plane spanning double stars that can be packed into complete geometric graphs of random point sets, as well as an upper bound for the smallest number of plane double stars that can be packed into any complete geometric graph.

In Chapter 5 we use the results from Chapters 3 and 4 to construct packings for special point sets, namely Horton point sets and point sets with many halving lines.

The third part of the thesis contains the results that are not about double stars. In Chapter 6 we discuss partitions into plane spanning paths. We show that for point sets with exactly one point not on the boundary of the convex hull, the induced complete geometric graph can be partitioned into plane spanning paths if and only if the point set is crossing-dominated by a point set in convex position. We also give an example that this is not true for point sets with more than one point not on the boundary of the convex hull.

In Chapter 7 we consider the more general problem of coloring arrangements of line segments in such a way, that we do not get any monochromatic crossings. We show that it is $\mathcal{NP}$-complete to decide whether a given partial coloring of a line segment arrangement with a fixed number of at least 3 colors can be extended to a complete coloring of the arrangement without monochromatic crossings.

Chapter 2

Survey of Related Results

2.1 Plane Spanning Trees in Complete Geometric Graphs

In the paper where the question motivating this thesis is first posed, the authors give a sufficient condition for the existence of a partition of a complete geometric graph into plane spanning trees:

Theorem 2.1 ([6]) *Let $\mathcal{P}$ be a point set with $n = 2m$ points. Suppose that there is a set $\mathcal{L}$ of pairwise non-parallel lines with exactly one point of $\mathcal{P}$ in each open unbounded region formed by $\mathcal{L}$. Then $K(\mathcal{P})$ can be partitioned into plane spanning double stars.*

If $\mathcal{P}$ has an even number of at most 8 points, then $K(\mathcal{P})$ can always be partitioned into plane spanning trees [2]. To show that the same holds for larger point sets, it would be sufficient to find a partition of $K(\mathcal{P})$ into plane spanning trees for all crossing-maximal point sets $\mathcal{P}$, as shown by Pilz and Welzl [17].

The case where the point set is in convex position is well understood. Two graphs drawn on a point set in convex position are called *convex isomorphic* if the underlying graphs are isomorphic and the clockwise ordering of the vertices is preserved under this isomorphism. Every complete geometric graph drawn on a point set of even size in convex position allows partitions into plane spanning trees, and these partitions can be characterized as follows:

Theorem 2.2 ([6]) *Let $\mathcal{P}$ be a point set with $n = 2m$ points in convex position. Let $T_1, \ldots, T_m$ be a partition of $K(\mathcal{P})$ into plane spanning trees. Then $T_1, \ldots, T_m$ are symmetric caterpillars that are pairwise convex isomorphic. Conversely, for any symmetric caterpillar T on n vertices, $K(\mathcal{P})$ can be partitioned into m plane spanning copies of T that are pairwise convex isomorphic.*

This in particular means that for an even number of points in convex position, the induced complete geometric graph even allows a partition into plane spanning paths. This is not true for general point sets, as shown by Aicholzer et al. [2] On the other hand they prove that at least 2 plane spanning paths can be packed into any complete geometric graph with at least 4 vertices. They also show a packing result for general plane spanning trees:

Theorem 2.3 ([2]) *Let $\mathcal{P}$ be a point set with n points. Then $K(\mathcal{P})$ allows a packing with $\sqrt{\frac{n}{12}}$ plane spanning trees.*

The spanning trees that they construct are double stars, but they also give a construction for plane spanning trees with lower maximum degree:

Theorem 2.4 ([2]) *Let $\mathcal{P}$ be a point set with n points and $k \leq \sqrt{\frac{n}{12}}$. Then $K(\mathcal{P})$ allows a packing with k plane spanning trees such that the maximum degree of any tree is in $\mathcal{O}(k^2)$. Also, the diameter of each tree is in $\mathcal{O}(\log(\frac{n}{k^2}))$.*

Bose et al. prove a result for coverings, that is similar to Theorem 2.3:

Theorem 2.5 ([6]) *Let $\mathcal{P}$ be a point set with n points in convex position. Then $K(\mathcal{P})$ can be covered with $n - \sqrt{\frac{n}{12}}$ plane spanning trees.*

All these results use the fact that in every complete geometric graph, there is a set of at least $\sqrt{\frac{n}{12}}$ pairwise crossing edges [4], called a *crossing family*. The actual size of the largest crossing family in any complete geometric graph is conjectured to be significantly larger, and a proof of the existence of larger crossing families would immediately improve the above theorems about packings and coverings.

2.2 Plane Spanning Trees in General Geometric Graphs

While it is easy to see that every complete geometric graph contains a plane spanning tree, there are geometric graphs that do not contain one. An easy example for such a geometric graph is a straight-line drawing of a tree that has crossings. A partition of a complete geometric graph into plane spanning trees can be thought of as a process of successively taking away plane spanning trees from a geometric graph, such that the remaining graph still contains a plane spanning tree. Thus, conditions for the existence of plane spanning trees in geometric graphs might be of interest when trying to find partitions. Rivera-Campo and Urrutia-Galicia [19], as well as Rivera-Campo [18] give sufficient conditions.

Theorem 2.6 ([19]) *Let G be a geometric graph with $n \geq 3$ vertices with vertex set $\mathcal{P}$. Let k be the number of empty triangles for which the induced subgraph of G is not connected. If $k \leq n - 3$, then G has a plane spanning tree.*

Theorem 2.7 ([18]) *Let G be a geometric graph with $n \geq 5$ vertices. If every subgraph of G induced by 5 vertices has a plane spanning tree, then G has a plane spanning tree.*

Rivera-Campo [18] also conjectured that if $G \setminus v$ has a plane spanning tree for any vertex v of G, then G has a plane spanning tree. But Schüler and Spillner [20] gave a counterexample to this claim.

On the other hand, Keller et al. [14] gave a necessary condition for the existence of plane spanning trees in geometric graphs by characterizing the smallest geometric graphs whose complements contain no plane spanning trees.

Definition 2.8 *Let $\mathcal{P}$ be a point set. A plane spanning subgraph C of $K(\mathcal{P})$ is a* comb *of $K(\mathcal{P})$ if:*

1. *The intersection of C with the boundary of $Conv(\mathcal{P})$ is a plane path S and possibly some isolated vertices.*
2. *Each vertex that is not in S is connected by a unique edge to an interior vertex of S.*
3. *For each edge e of C, the line ℓ_e spanned by e does not cross any edge of C.*

See Figure 2.1 for an example of a comb. Note that a comb is a caterpillar with spine $S \setminus \{v, w\}$, where v and w are the endpoints of S.

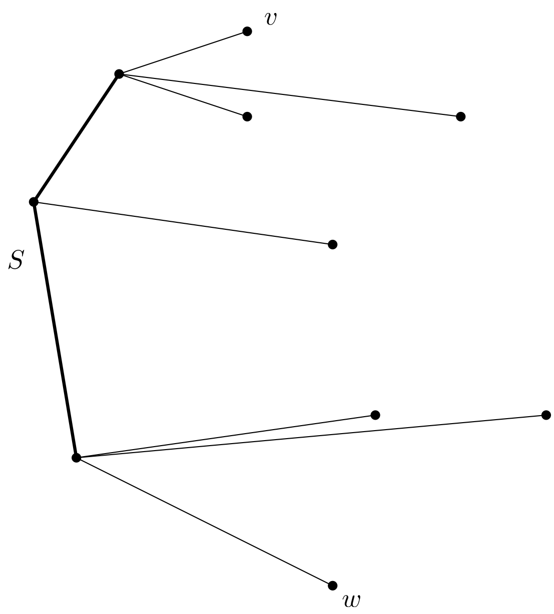


Figure 2.1: A comb of a complete geometric graph with 10 vertices.

We say that a geometric graph B, drawn on a point set $\mathcal{P}$, *blocks* the family of plane spanning trees if it has at least one edge in common with each plane spanning tree of $K(\mathcal{P})$. We call B a *minimal blocker* if it has the smallest number of edges among all graphs that block the family of plane spanning trees.

Theorem 2.9 ([14]) *A geometric graph B drawn on a point set $\mathcal{P}$ is a minimal blocker if and only if it is either a spanning star or a comb of $K(\mathcal{P})$.*

Corollary 2.10 *Let G be a geometric graph drawn on a point set $\mathcal{P}$. If the complement $\overline{G}$ of G contains a comb of $K(\mathcal{P})$ or a spanning star, then G does not contain a plane spanning tree.*

Note that a comb, as well as a spanning star, is a plane spanning tree, so if $\overline{G}$ contains a comb of $K(\mathcal{P})$ or a spanning star, then $\overline{G}$ contains a plane spanning tree. Károlyi, Pach and Tóth [13] have shown that this is always true if G does not contain a plane spanning tree.

Theorem 2.11 ([13]) *If the edges of a complete geometric graph are colored arbitrarily with two colors, then there exists a monochromatic plane spanning tree.*

In general, deciding whether a geometric graph contains a plane spanning tree is $\mathcal{NP}$-complete, as shown by Jansen and Woeginger [12].

2.3 Other Plane Subgraphs

Apart from spanning trees, there are other plane subgraphs that can be contained in a geometric graph or that we can use to construct packings or coverings. There are results of these types for example for general trees, triangulations or perfect matchings. Aichholzer et al. [1] have considered the following question: Given a complete geometric graph, how many edges can be removed such that the remaining graph still contains a certain plane subgraph? They prove results for the types of subgraphs mentioned above.

Theorem 2.12 ([1]) *For $2 \leq k \leq n-1$, for every complete geometric graph G on n vertices, and for every subgraph H of G with at most $\lceil \frac{kn}{2} \rceil - 1$ edges, the geometric graph $G \setminus H$ contains a plane tree that spans $n-k+1$ vertices.*

This theorem is tight with respect to the number of edges in H.

Theorem 2.13 ([1]) *Let $\mathcal{P}$ be a point set of $n \geq 3$ points in convex position. Let A be the set of interior edges of $K(\mathcal{P})$. Let H be a subgraph of $K(\mathcal{P})$ consisting of at most $n-3$ edges of A. Then $G \setminus H$ contains a triangulation.*

Again the theorem is tight with respect to the number of edges in H. The edges that are not in A can never be removed as they appear in every triangulation. If the point set is not required to be in convex position, then there might be less edges that can be removed. As with plane spanning trees, it is $\mathcal{NP}$-complete to decide whether a given geometric graph contains a triangulation of its vertex set, which was shown by Lloyd [16].

Theorem 2.14 ([1]) *For every complete geometric graph G on $n = 2m$ vertices, and for every subgraph H of G with at most m vertices in each component, the geometric graph $G \setminus H$ contains a plane perfect matching.*

For plane perfect matchings, Biniaz, Bose, Maheshwari and Smid [5] have also investigated the packing problem and they have found upper and lower bounds.

Theorem 2.15 ([5]) *There exist point sets $\mathcal{P}$ with $n = 2m \geq 6$ points such that no more than $\lceil \frac{n}{3} \rceil$ plane perfect matchings can be packed into $K(\mathcal{P})$.*

Theorem 2.16 ([5]) *Let $\mathcal{P}$ be a point set with $n = 2m$ points. Then at least $\lceil \log_2 n \rceil - 2$ plane perfect matchings can be packed into $K(\mathcal{P})$.*

As every perfect matching consists of $\frac{n}{2}$ edges, no more than $n - 1$ plane perfect matchings can be packed into a complete geometric graph. Biniaz et al. [5] prove that the maximum number of perfect matchings that can be packed into a complete geometric graph is between $\frac{n}{2}$ and $n - 1$, where the lower bound is attained for a complete geometric graph drawn on a point set in convex position. We are able to improve this result and show that the upper bound $n - 1$ is tight.

Remark 2.17 *There exist point sets $\mathcal{P}$ with $n = 2m$ points such that $K(\mathcal{P})$ can be partitioned into plane perfect matchings.*

Proof Let $\mathcal{P}'$ be a point set of $n - 1$ points in convex position. As n is even, $n - 1$ is odd and we can construct $n - 1$ pairwise edge-disjoint plane matchings on $K(\mathcal{P}')$ such that for every point p in $\mathcal{P}'$ there is exactly one matching where p has degree 0. Place $\mathcal{P}'$ on a semi-circle and place a last point q such that q sees every point of $\mathcal{P}'$ in $K(\mathcal{P}' \cup \{q\})$ (See Figure 2.2). For every $p \in \mathcal{P}'$, add the edge (p, q) to the unique matching where p has degree 0. This gives $n - 1$ pairwise edge-disjoint plane perfect matchings on a point set with n points. $\square$

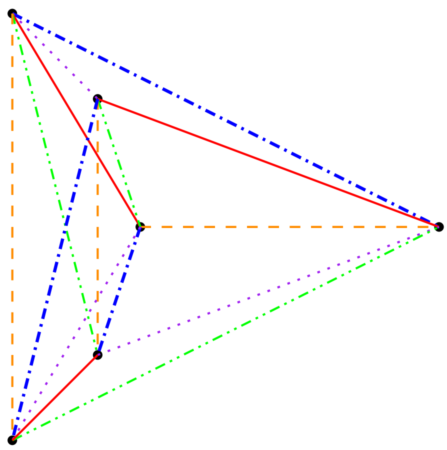


Figure 2.2: A complete geometric graph on 6 vertices that allows a partition into plane perfect matchings.

2.4 Edge-coloring Geometric Graphs

A partition of a geometric graph with an even number n of vertices into plane spanning trees induces a coloring of its edges with $\frac{n}{2}$ colors that has no monochromatic crossings. If the vertices are in convex position, then the complete geometric graph has a set of $\frac{n}{2}$ pairwise crossing edges, thus any such coloring requires at least $\frac{n}{2}$ colors. On the other hand, there are complete geometric graphs whose edge sets can be colored with $\lceil \frac{n}{4} \rceil$ colors without getting monochromatic crossings [7]. It is still an open question whether every complete geometric graph can be colored with $\frac{n}{2}$ colors without monochromatic crossings.

A different way of stating this problem is to ask for the chromatic number of the *interior intersection graph* of a complete geometric graph. The interior intersection graph I of a geometric graph G is constructed by defining a vertex v_e for each segment e in G, and defining an edge between two vertices v_e and v_f in I if the corresponding segments e and f in G cross. If we also define edges between vertices in I corresponding to incident segments in G, we get the so called *intersection graph* of a geometric graph. Araujo et al. [3] have shown that the chromatic number of the intersection graph of a complete geometric graph on n vertices lies between n and $cn^{\frac{3}{2}}$ for some constant $c > 0$. Note that a lower bound of $n - 1$ is trivial, as each vertex of a complete geometric graph is incident to $n - 1$ edges.

By slightly shrinking each segment in G, we can turn the problem of determining the chromatic number of the interior intersection graph of a complete geometric graph G into a problem of finding the chromatic number of the intersection graph of a line segment arrangement with distinct endpoints. The chromatic number of the intersection graph of line segment arrangements is in general not bounded by the size of the largest clique, but Fox and Pach [10] have shown that it cannot be arbitrarily large, even for a more general setting. They consider graphs G that are K_k-free intersection graph of convex sets in the plane, where K_k-free means that G does not contain a clique of size k. For any such graph G, the bound the chromatic number $\chi(G)$ of G.

Theorem 2.18 ([10]) *If G is a K_k-free intersection graph of n convex sets in the plane, then*

$$\chi(G) \leq \left(c\frac{\log n}{\log k}\right)^{13 \log k},$$

where c is an absolute constant.

However, determining the chromatic number of the intersection graph of a line segment arrangement is $\mathcal{NP}$-complete, as shown by Ehrlich, Even and Tarjan [9].

Part II

Spanning Double Stars

Chapter 3

Partitions into Plane Spanning Double Stars

In this chapter, we will show that for any partition of a complete geometric graph into plane spanning double stars, the set of the spines of the double stars forms a perfect matching, called the spine matching. In order for a perfect matching to be a spine matching, the matching has to fulfill certain conditions. We discuss a necessary condition, a sufficient condition, and we show that we can check in polynomial time whether a given perfect matching is the spine matching of a partition of a complete geometric graph into plane spanning double stars.

We start with a few observations that hold for any partition of an (abstract) complete graph into spanning double stars. Consider a complete graph K_n, where n is even, and assume that it is partitioned into $\frac{n}{2}$ spanning double stars. Let M be the set of the spines of the double stars. Note that $|M| = \frac{n}{2}$.

Lemma 3.1 *The set of spines M of a partition of K_n into spanning double stars is a perfect matching.*

We will call this perfect matching the *spine matching*.

Proof We want to show that no two edges of M are incident. Assume for the sake of contradiction that two edges $e = (p, q)$ and $f = (p, r)$ share an endpoint p. Let E and F be the spanning double stars with spines e and f, respectively. Consider the edge $g = (q, r)$. As all double stars in the partition must be spanning, the point r must be connected to the edge e, which means that $f \in E$ or $g \in E$. As f is already the spine of F, we conclude that $g \in E$. On the other hand q must also be connected to the edge f and with the same argument we conclude $g \in F$, which is a contradiction. □

Lemma 3.2 *Let K_n be partitioned into spanning double stars. Then all double stars in the partition are symmetric.*

Proof By Lemma 3.1, any vertex v of K_n is a leaf of $\frac{n}{2} - 1$ double stars and incident to the spine of exactly one double star D. Thus, as every edge in K_n belongs to some double star, the degree of v in D must be $n - 1 - (\frac{n}{2} - 1) = \frac{n}{2}$. □

Actually, we could have also proved Lemma 3.1 with an easy degree argument: If two spines in M were incident, then there would be a vertex v that is not incident to any spine. Thus v is a leaf in every spanning double star in the partition and therefore has degree $\frac{n}{2}$, which is a contradiction, as we know that v has degree $n - 1$. However, the proof above will be handy when we discuss packings with plane spanning double stars in Chapter 4.

Combining the two lemmas, we get the following result:

Corollary 3.3 *Let K_n be partitioned into spanning double stars and let V' be the vertices of any subset of the spine matching M. Then the induced subgraph on V' inherits a partition into symmetric spanning double stars.*

Proof Color each double star in the partition with a different color, including red. Now delete the vertices incident to the red spine and consider the colored subgraph induced by the remaining vertices. Clearly, this subgraph contains no red edges, as each red edge is incident to the red spine. Also, all deleted edges that are not red must be leaf edges, as we know from Lemma 3.1 that no two spines are incident. Thus the remaining graph is still partitioned into plane spanning double stars, and by Lemma 3.2 all double stars in the partition are symmetric. The result follows by induction. □

For abstract complete graphs, every perfect matching M is a spine matching of a partition into spanning double stars: Let each edge in M have a different color. We want to color the remaining edges of the complete graph in such a way that each color class is a spanning double star with spine in M. Consider the K_4 induced by two edges in M, colored red and blue. Then the remaining edges in the K_4 have to be colored red and blue as well. Fixing the color of one of the remaining edges of the induced K_4 already determines the colors of all edges (see Figure 3.1). Thus, for any two edges in M, there are exactly two ways to color the remaining edges in the induced K_4. Doing this for every pair of edges in M, we get a coloring of the complete graph and it is easy to see that each color class is indeed a spanning double star. As there are two possibilities to color the remaining edges of the K_4 induced by any two edges in M, we conclude that any perfect matching M with m edges is the spine matching of $2^{\binom{m}{2}}$ different partitions into spanning double stars.

However, for complete geometric graphs, we also want every spanning double star in the partition to be plane. Unfortunately, there are geometric matchings for which all of the $2^{\binom{m}{2}}$ possible partitions into spanning double stars contain at least one crossing double star. This motivates the following

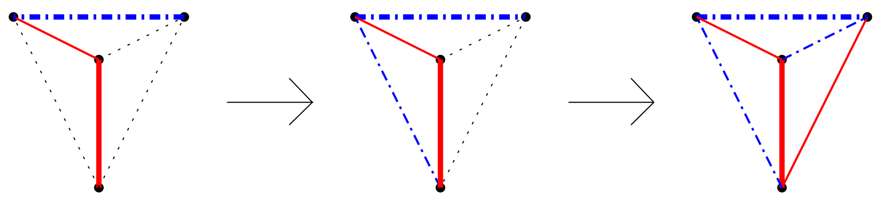


Figure 3.1: Choosing a color for one of the remaining edges determines the color of the other three. The edges in M are drawn thick.

definition: Let M be a geometric matching and $\mathcal{P}$ its set of vertices. We call the matching M *expandable* if it is the spine matching of a partition of $K(\mathcal{P})$ into plane spanning double stars. In order to find conditions for a matching to be expandable, we will now also consider the geometrical properties of a geometric graph.

Let e be an edge between two points p and q. The *supporting line* ℓ_e of e is the line through p and q.

Let e and f be two edges and let s be the intersection of their supporting lines. If s lies in both e and f, we say that e and f *cross*. If s lies in f but not in e, we say that e *stabs* f and we call the vertex of e that is closer to s the *stabbing vertex* of e. If s lies neither in e nor in f, or even at infinity, we say that e and f are *parallel*. See Figure 3.2 for an illustration.

Note that our notion of parallel is not transitive. Also note that the intersection s of the supporting lines of two non-incident edges never coincides with a point in the point set, as we assume the point set to be in general position.

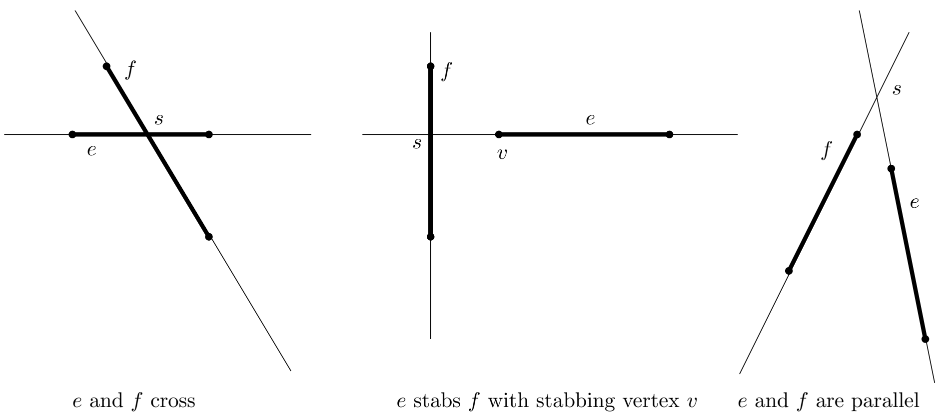


Figure 3.2: Crossing, stabbing and parallel edges

Lemma 3.4 *A (geometric) matching M consisting of two edges a and b is expandable if and only if a and b are not parallel.*

Proof First assume that a and b are parallel. We show that then M is not expandable.

Let $a = (p,q)$ be red and $b = (r,s)$ blue. As a and b are parallel, the points p, q, r and s form a convex quadrilateral, implying that $K(\{p,q,r,s\})$ has a crossing. We try to construct a partition of $K(\{p,q,r,s\})$ into plane spanning double stars. Assume without loss of generality that the edges (p,s) and (q,r) cross. Then they cannot have the same color, so without loss of generality let (p,s) be red and let (q,r) be blue. Then the edge (q,s) cannot be red as there would be a red triangle otherwise. By the same argument the edge (q,s) also cannot be blue. Thus $K(\{p,q,r,s\})$ cannot be partitioned into plane spanning double stars.

Now assume that a and b are not parallel. Then they are either stabbing or crossing. In both cases M is expandable, as can be seen in Figure 3.3. $\square$

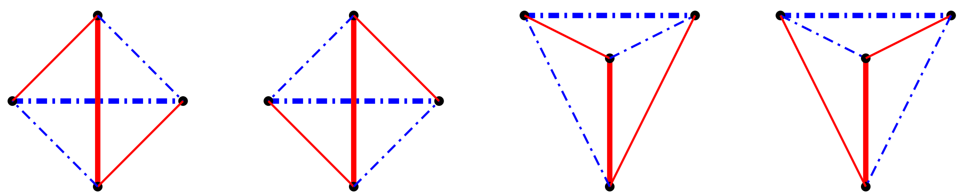


Figure 3.3: Any pair of crossing or stabbing edges is expandable. The spines are drawn thick.

In Figure 3.3 we also see that a matching consisting of two edges that are stabbing or crossing is the spine matching of two different partitions into plane spanning double stars.

For any edge $e = (p,q)$ in a double star D with spine (q,r), we say that e is a *left edge (of D)* if the ordered triple (r,q,p) encodes a left turn. Otherwise we call e a *right edge (of D)*. Consider a partition of a geometric K_4 into plane spanning double stars with spines e and f. We say that the pair $\{e,f\}$ is *left-oriented* if there are more left edges than right edges in the partition. If there are more right edges than left edges, we call the pair *right-oriented*. Given two edges a and b in any matching M that are either stabbing or crossing, we say that we *left-orient* (*right-orient*) the pair $\{a,b\}$ if we partition the induced K_4 in such a way that the pair $\{a,b\}$ is the spine matching and the pair $\{a,b\}$ is left-oriented (right-oriented).

Note that in Figure 3.3 the first and third partition have a left-oriented spine matching and the second and fourth partition have a right-oriented spine matching. Also note that for the left-oriented crossing spines, all remaining edges are left edges, whereas for the left-oriented stabbing spines there is a right edge, and symmetrically for the right-oriented partitions. The following lemma makes this a bit more precise.

Lemma 3.5 *Consider a partition of a geometric K_4 into plane spanning double stars with spines a, colored red, and b, colored blue, and assume that the pair $\{a, b\}$ is left-oriented (right-oriented). Let E be the set of edges in the K_4 apart from a and b. There is a blue right (left) edge in E if and only if a stabs b. Also, if a stabs b, then the blue right (left) edge is incident to the stabbing vertex of a.*

Proof We will proof this lemma for the left-oriented case. The right-oriented case is symmetrical. By Lemma 3.4, a and b are either stabbing or crossing and therefore the pair $\{a, b\}$ can indeed be left-oriented.

If a stabs b, it is clear from Figure 3.3 that there is a blue right edge that is incident to the stabbing vertex of a. If however a does not stab b, then either b stabs a or a and b cross. If b stabs a, then the only right edge in E is red. If a and b cross, all edges in E are left edges. □

3.1 A necessary condition

We have already seen that a matching consisting of two edges is expandable if and only if the two edges are not parallel. For larger matchings, the situation is more complicated, but we can still find some configurations that cannot occur in the matching. See Figure 3.4 for a drawing of these configurations.

A *cross-blocker* is a triple $C = \{e, f, g\}$ of three pairwise non-incident edges such that e and f cross, g stabs both e and f, g does not intersect the convex hull of e and f, and both vertices of g see only one vertex p of e and one vertex q of f in C.

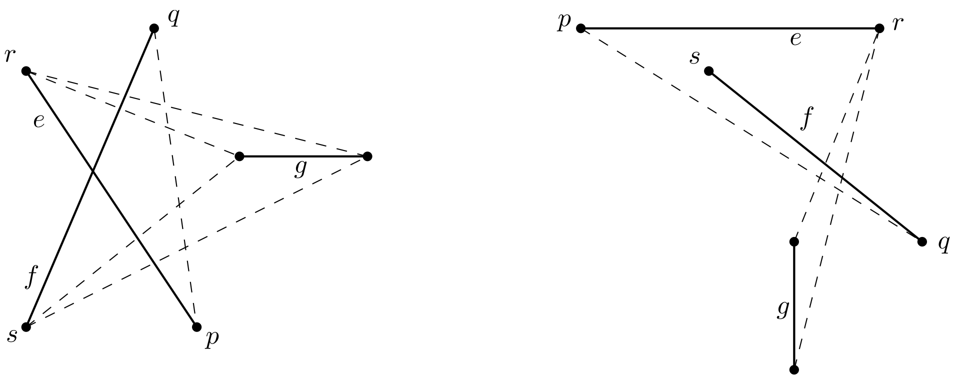
A *stab-blocker* is a triple $S = \{e, f, g\}$ of three pairwise non-incident edges such that f stabs e, g stabs both f and e, g does not intersect the convex hull of e and f, and both vertices of g see only one vertex p of e in S.

Lemma 3.6 *Let M be a cross-blocker or a stab-blocker. Then M is not expandable.*

Proof We start by stating the following easy observation: Let $\mathcal{P}$ be a point set, and let c be a point outside the convex hull of $\mathcal{P}$ with the property that c only sees two vertices a and b of the convex hull of $\mathcal{P}$. Then every edge incident to c that intersects the convex hull of $\mathcal{P}$ crosses the edge (a, b).

Assume first that $M = \{e, f, g\}$ is a cross-blocker. Let e be red, f blue and g green. Let p be the vertex of e that is seen by both vertices of g, and let q be the vertex of f that is seen by both vertices of g. Let r be the vertex of e that is not seen by the vertices of g and let s be the vertex of f that is not seen by the vertices of g. Both vertices of g see only two vertices of the convex hull of e and f, namely p and q. Consider the two edges between r and the vertices of g. One of these edges has to be green and the other one has to be red. By the observation above, both these edges cross the edge (p, q), implying that (p, q) can be neither red nor green. Applying the same argument to the edges between s and the vertices of g, we also see that (p, q) can be neither blue nor green. But (p, q) has to be red or blue, and we conclude that M is not expandable.

Now assume that $M = \{e, f, g\}$ is a stab-blocker. Let e be red, f blue and g green. Let p be the vertex of e that is seen by both vertices of g and let q be the vertex of f that lies on the boundary of the convex hull of e and f. Let r be the vertex of e that is not seen by the vertices of g and let s be the vertex of f that lies inside the convex hull of f and g. Again, both vertices of g see only two vertices of the convex hull of e and f, namely p and q and by considering the edges between r and the vertices of g, as well as the edges between s and the vertices of g, we can again see that the edge (p, q) can be neither green, red nor blue. But again (p, q) has to be red or blue, and we conclude that M is not expandable. $\square$



a cross-blocker a stab-blocker

Figure 3.4: A cross-blocker (left) and a stab-blocker(right)

We are now ready to prove the main theorem of this section:

Theorem 3.7 *Let $K(\mathcal{P})$ be partitioned into plane spanning double stars. Then the corresponding spine matching M*

- *does not contain two parallel edges,*
- *does not contain a cross-blocker and*
- *does not contain a stab-blocker.*

Proof By Lemma 3.4 and Lemma 3.6, none of the three configurations is expandable. But, by Corollary 3.3, a partition of $K(\mathcal{P})$ into plane spanning double stars would induce a partition of the induced subgraph of the configuration into plane spanning double stars. $\square$

This allows us to construct a point set whose complete geometric graph cannot be partitioned into plane spanning double stars. For every $k > 0$, we define the *bumpy wheel set* BW_k as follows:

Place $k-1$ points in convex position and partition them into three sets A_1, A_2, A_3 of consecutive points such that $||A_i| - |A_j|| \leq 1$, $i \neq j$. Let H_i, be the convex hull of $\cup_{j \neq i} A_j$. Place the last point p in the interior such that it lies outside of H_i for all $i \in \{1, 2, 3\}$. See Figure 3.5 for a depiction of BW_{10}.

Theorem 3.8 *For every $k \geq 9$, the complete geometric graph $K(BW_k)$ cannot be partitioned into plane spanning double stars.*

Proof If k is odd, then it is clear that $K(BW_k)$ cannot be partitioned into plane spanning double stars. So assume that k is even, and thus $k \geq 10$.

Consider any perfect matching M on BW_k and assume for the sake of contradiction that it is expandable. We can assume without loss of generality that the interior point p is matched with a point in A_1 by an edge e. We claim that there are at least two edges between points in $A_2 \cup A_3$. Any point in $A_2 \cup A_3$ that is not matched with another point in $A_2 \cup A_3$ must be matched with a point in A_1. Thus the number of points in $A_2 \cup A_3$ that are matched with another point in $A_2 \cup A_3$ is at least $|A_2| + |A_3| - (|A_1| - 1)$. We want to show that $|A_2| + |A_3| - (|A_1| - 1) \geq 4$, or equivalently $|A_2| + |A_3| - |A_1| \geq 3$. Assume without loss of generality that A_3 does not have more points than A_2. If $|A_2| \geq 4$, we have $|A_2| + |A_3| - |A_1| \geq |A_2| - 1 \geq 3$. If however $|A_2| < 4$, then, as we assumed that A_3 does not have more points than A_2, we must have $k = 10$. Thus in this case we have $|A_1| = |A_2| = |A_3| = 3$ and thus $|A_2| + |A_3| - |A_1| = 3$.

Hence, there are at least two edges between points in $A_2 \cup A_3$. As the set $A_2 \cup A_3$ is in convex position, these two edges are either parallel, or they cross. If they are parallel or one of them is parallel to e, we get a contradiction to Theorem 3.7. However, if they cross and neither of them is parallel to e, then those three edges form a cross-blocker, which is again a contradiction to Theorem 3.7. Thus M is not expandable. As this is true for any perfect matching on BW_k, we deduce that $K(BW_k)$ cannot be partitioned into plane spanning double stars. $\square$

3.2 A sufficient condition

We will show that if a perfect matching on a point set $\mathcal{P}$ satisfies certain conditions, then it can be expanded to a partition of $K(\mathcal{P})$ into plane spanning double stars. But first we start with some definitions and a preliminary lemma:

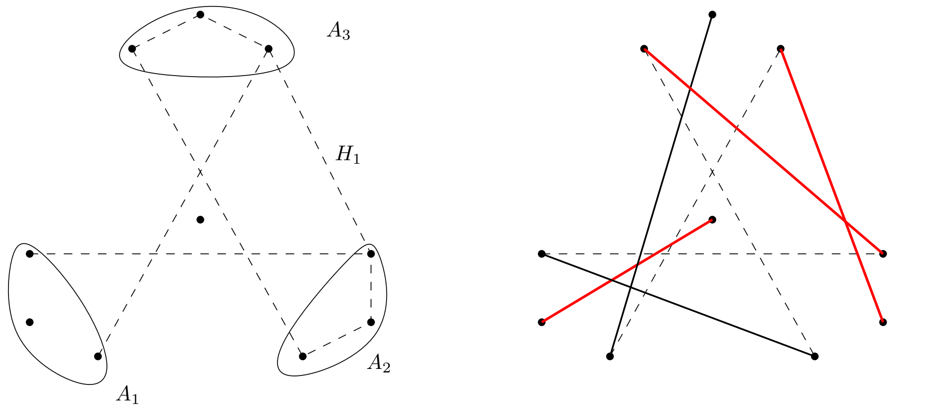


Figure 3.5: The point set BW_{10} (left) and a cross blocker in a matching on this point set (right)

Lemma 3.9 (Cross-Stab-Lemma) *Consider three edges, e, f and g, where f and g cross and f stabs e with stabbing vertex v. Let C be the convex hull of e and g. Then $v \in C$.*

Proof Let a be the intersection of ℓ_f and ℓ_g and let b be the intersection of ℓ_f and ℓ_e. Note that a lies on f and g and b lies on e. The stabbing vertex v must lie between a and b, and as a and b are in C, it follows that also v is in C. □

A *stabbing chain* are three edges, e, f and g, where e stabs f and f stabs g. We call f the middle edge of the stabbing chain. If also g stabs e we call the three edges a *stabbing cycle*.

See Figure 3.6 for a drawing of some stabbing chains. Note that a stabbing cycle can be seen as three stabbing chains where each edge is the middle edge in one of the stabbing chains.

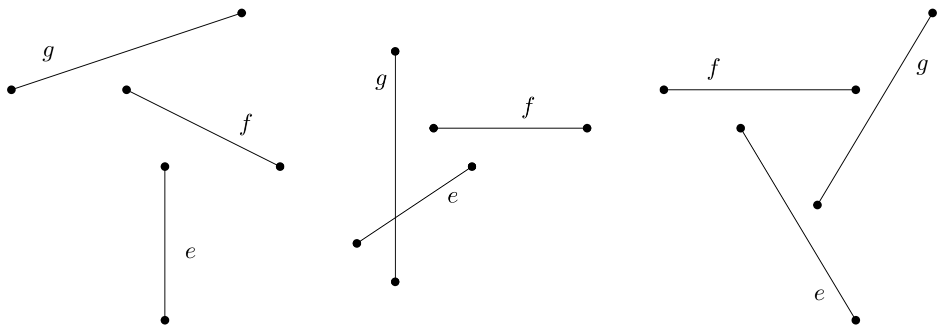


Figure 3.6: Two different stabbing chains (left and middle) and a stabbing cycle (right)

We are now ready to state the sufficient condition:

Theorem 3.10 *Let $\mathcal{P}$ be a point set. If there exists a perfect matching M on $\mathcal{P}$, such that*

(a) *no two edges are parallel,*

(b) *if an edge e stabs two other edges f and g, then the respective stabbing vertices of e lie inside the convex hull of f and g, and*

(c) *if there is a stabbing chain, then the stabbing vertex of the middle edge lies inside the convex hull of the other two edges,*

then M is expandable.

Note that a stab-blocker is a stabbing chain that satisfies condition (c), but not (b).

The theorem follows immediately from the following lemma:

Lemma 3.11 *Let $\mathcal{P}$ be a point set and let M be a perfect matching on $\mathcal{P}$ that satisfies (a), (b) and (c). Then left-orienting each pair of edges in M induces a partition of $K(\mathcal{P})$ into plane spanning double stars.*

Proof As no two edges in the matching M are parallel, we can indeed left-orient each pair and this induces a partition of $K(\mathcal{P})$ into spanning double stars, where M is the spine matching. It remains to show that all the double stars are plane. Assume for the sake of contradiction that there is a red double star with spine $e = (p, q)$ that has two crossing edges (p, r) and (q, s). Then one of the edges has to be a left edge and the other one has to be a right edge. Assume without loss of generality that (p, r) is a left edge and (q, s) is a right edge.

Both r and s are incident to a spine in M. If they are incident to the same spine, then, as p, q, r and s form a convex quadrilateral, this spine is parallel to e, which is a contradiction to condition (a). So assume that s is incident to a blue spine f and r is incident to a green spine $g = (r, t)$. As (q, s) is a right edge, by Lemma 3.5 f must stab e with stabbing vertex s. As (p, r) is a left edge, again by Lemma 3.5, g cannot stab e with stabbing vertex r. However, g might stab e with stabbing vertex t. Also, by condition (a), e and g cannot be parallel. We distinguish the different remaining cases for the spines e and g. See Figure 3.7 for an illustration of the cases.

Case 1: e and g cross. Let H be the convex hull of e and g. If e and g cross, then (p, r) bounds H. As (p, r) and (q, s) cross, the line through p and r separates the points q and s. As q is incident to e, q lies in H and thus s cannot lie in H. Consider the spines f and g. If they are parallel, we get a contradiction to condition (a). If f stabs g, then by condition (b) the point s must be in H, so we again get a contradiction. If g stabs f, then f is the middle edge of the stabbing chain defined by g, f and e, and by condition (c), s must lie in H, which is a contradiction. Finally, if f and g cross, then by the Cross-Stab-Lemma s must lie in H, which is again a contradiction.

Case 2: e stabs g with stabbing vertex q. Let H be the convex hull of e and g. If e stabs g with stabbing vertex q, then H is a triangle with vertices p, r and t. Thus again (p,r) bounds H, and as q is in H we can again deduce that s is not in H. We can now continue analogous to case 1.

Case 3: e stabs g with stabbing vertex p. Consider the convex hull of f and g and denote it by H'. We claim that p cannot lie in H'. Let ℓ_g be the supporting line of g. If ℓ_g separates p and s, then f stabs g with stabbing vertex s and thus g bounds H'. As s is in H', p cannot be in H'. If ℓ_g does not separate p and s, then either f and g cross or f stabs g but the stabbing vertex is not s. In both cases the edge (r,s) bounds H'. Also the line through r and s separates p and t. As t is in H' we conclude that p is not in H'. So p indeed cannot lie in H'. But e is the middle edge of the stabbing chain defined by f, e and g, so by condition (c), p must lie in H'. This is a contradiction.

Case 4: g stabs e with stabbing vertex t. Let H be the convex hull of e and g. If g stabs e with stabbing vertex t, H is a triangle with vertices p, r and q. Thus again (p,r) bounds H, and as q is in H we can again deduce that s is not in H and we can again continue analogous to case 1.

We have thus proven by contradiction that each double star in the partition is indeed plane. □

Using Theorem 3.10 we can reprove a sufficient condition for the existence of a partition into plane spanning double stars from Bose et al. [6].

Theorem 3.12 ([6]) *Let $\mathcal{P}$ be a point set with an even number n of points. Suppose that there is a set $\mathcal{L}$ of $\frac{n}{2}$ pairwise non-parallel lines with exactly one point of $\mathcal{P}$ in each open unbounded region formed by $\mathcal{L}$. Then $K(\mathcal{P})$ can be partitioned into plane spanning double stars.*

Proof Let C be a circle such that all points of $\mathcal{P}$ as well as all intersections of lines in $\mathcal{L}$ lie inside of C. The intersection points of C and the lines in $\mathcal{L}$ partition C into consecutive components $C_1, \ldots, C_n$, each corresponding to an unbounded region. Let p_i be the point in the unbounded region corresponding to C_i. For every $i \in \{1, \ldots, \frac{n}{2}\}$, match p_i with $p_{\frac{n}{2}+i}$. This induces a perfect matching M on $\mathcal{P}$. Note that each line in $\mathcal{L}$ is a halving line and that each edge in M intersects all lines in $\mathcal{L}$. We will show that this matching is expandable by proving that it satisfies the conditions (a), (b) and (c) of Theorem 3.10.

We start with condition (a). Pick two edges $e = (p,q)$ and $f = (r,s)$ from M and two lines ℓ_1 and ℓ_2 from $\mathcal{L}$, with the property that one of the endpoints of e and f lies in each unbounded region defined by ℓ_1 and ℓ_2. Assume

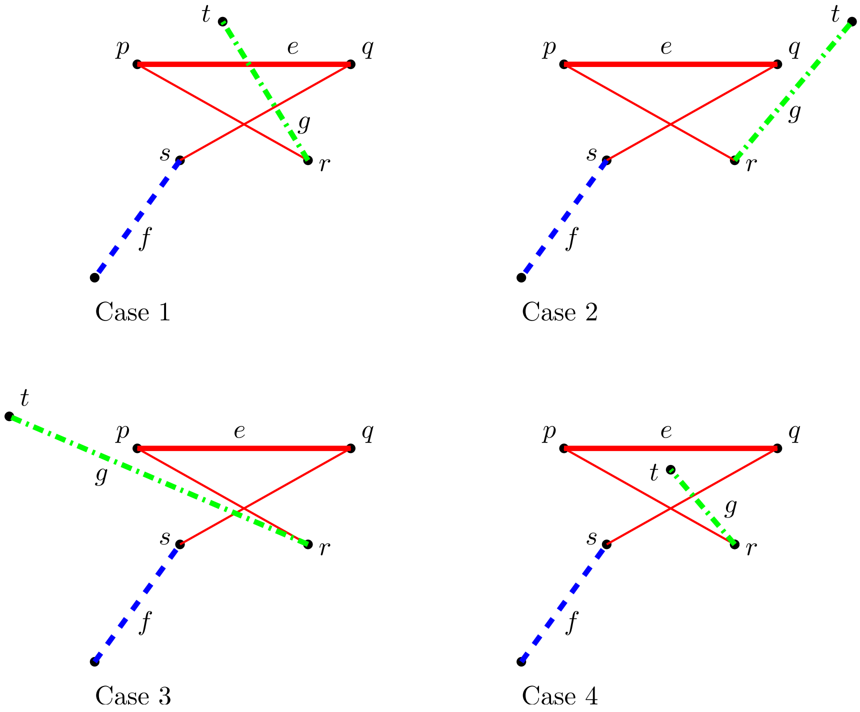


Figure 3.7: The cases in the proof of Lemma 3.11

without loss of generality that ℓ_1 is horizontal and ℓ_2 is vertical. Let p be in the bottom left region and let r be in the bottom right region. Then q is in the top right region and s is in the top left region. Assume without loss of generality that e intersects the top left region and let T_1 be the triangle bounded by e, ℓ_1 and ℓ_2. If s lies in T_1, then f stabs e. Assume without loss of generality that f intersects the top right region and let T_2 be the triangle bounded by f, ℓ_1 and ℓ_2. If q lies in T_2, then e stabs f. If s does not lie in T_1 and q does not lie in T_2, then e and f cross. Thus any two edges in M are either crossing or stabbing, and thus M satisfies condition (a) of Theorem 3.10.

Now we show that M satisfies conditions (b) and (c) by proving that for three edges e, f and g, with f stabbing e, the stabbing vertex of f lies in the convex hull of e and g. So, pick three edges $e = (p, q)$, $f = (r, s)$ and $g = (t, u)$ from M and three lines ℓ_1, ℓ_2 and ℓ_3 from $\mathcal{L}$, with the property that one of the endpoints of e, f and g lies in each unbounded region defined by ℓ_1, ℓ_2 and ℓ_3. Assume without loss of generality that f stabs e with stabbing vertex s. Let $A_1, \ldots, A_6$ be the unbounded regions defined by ℓ_1, ℓ_2 and ℓ_3, and assume without loss of generality that $s \in A_1$, $p \in A_2$, $t \in A_3$, $r \in A_4$, $q \in A_5$ and $u \in A_6$. Let ℓ_1 be the line separating A_1, A_2 and A_3 from A_4, A_5 and A_6. Let ℓ_2 be the line separating A_2, A_3 and A_4 from A_1, A_5 and A_6. Finally, let ℓ_3 be the line separating A_3, A_4 and A_5 from A_1, A_2 and A_6. See

Figure 3.8 for an illustration.

As f stabs e with stabbing vertex s, the edge e must intersect A_1. Let R be the part of A_1 that is bounded by e and note that s lies in R. Let H be the convex hull of e and g. We will show that $R \subset H$. Consider the edge (q, t). This edge does not intersect the line ℓ_3 as $q \in A_5$ and $t \in A_3$. Similarly, the edge (p, t) does not intersect the line ℓ_2. Let T be the triangle defined by the edges (q, t), (p, t) and $e = (p, q)$. As (q, t) does not cross ℓ_3 and (p, t) does not cross ℓ_2, we deduce that $R \subset T$. Clearly $T \subset H$. Thus we see that $R \subset H$ and as s lies in R, s also lies in H. Thus for any three edges e, f and g, with f stabbing e, the stabbing vertex of f lies inside the convex hull of e and g. This proves that M satisfies the conditions (b) and (c) of Theorem 3.10. □

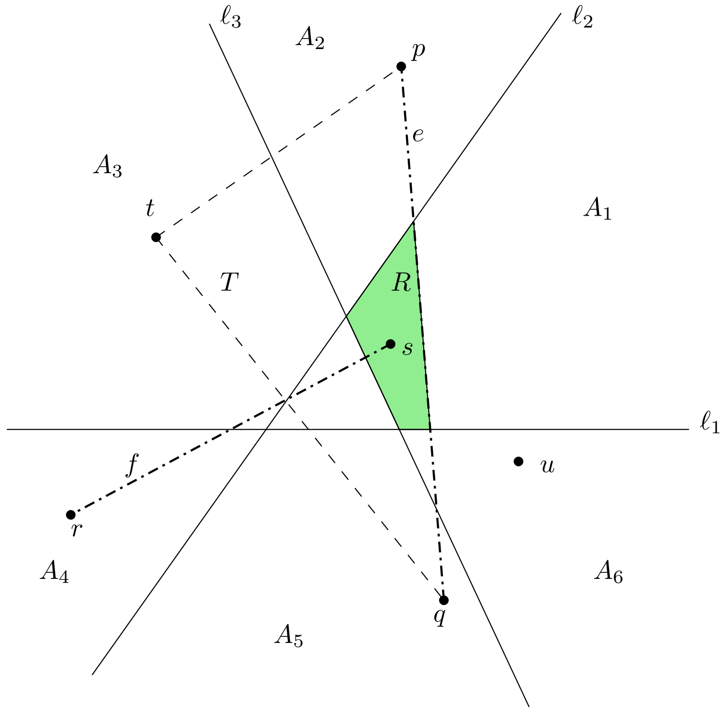


Figure 3.8: Illustration of the proof of Theorem 3.12

As we have seen in the last section, not all point sets allow an expandable perfect matching. However the requirement for a perfect matching to be parallel-free is not a big constraint.

Remark 3.13 *Every point set of even size allows a parallel-free perfect matching.*

Proof Let M be a perfect matching that maximizes the sum of the lengths of all edges. We claim that M is parallel-free. Assume for the sake of contradiction that two edges e and f in M are parallel. Then their endpoints form a convex quadrilateral Q. Delete e and f from the matching and instead insert the two crossing edges defined by Q. This gives us a new matching M' and by the triangle inequality the sum of the lengths of all edges in M' is higher than the sum of the lengths of all edges in M, which is a contradiction to the choice of M. □

3.3 Recognizing expandable matchings

So far in this chapter we have seen necessary and sufficient conditions for matchings to be expandable. In this section we will consider the decision problem where, given a perfect matching on a point set $\mathcal{P}$ in general position, we want to decide whether it is expandable. We will show that we can solve this problem in polynomial time.

Recall that for any two edges in a perfect matching, colored with two different colors, choosing a color for one of the four remaining edges of the induced K_4 already determines the color of all uncolored edges of this K_4. Thus there are exactly two possibilities for coloring the four remaining edges for each pair of edges in the matching. For the case where the two edges are not parallel, we called the two options "left-oriented" and "right-oriented". Expanding a parallel-free perfect matching to a partition into spanning double stars is thus just choosing for each pair of edges in the matching, whether the pair is left-oriented or right-oriented. The given perfect matching is then the spine matching of the partition.

Consider now the partition given by such a choice of orientation of each pair of spines in M, where M is parallel-free, and assume there is a monochromatic crossing, let us say of color red. Then, as M is parallel-free, the two crossing red edges a and b are incident to exactly three spines: both edges are incident to the red spine e, and each edge is incident to another spine, let us assume that a is incident to the blue spine f, and b is incident to the green spine g. The fact that both a and b are red already determines the orientation of the pairs $\{e, f\}$ and $\{e, g\}$, as a is part of the K_4 induced by e and f and b is part of the K_4 induced by e and g. Also, changing one or both orientations would give a partition where a and b have different colors. We call a set consisting of three spines e, f, g and two edges a, b a *potential monochromatic crossing* if a and b cross, a is incident to f, b is incident to g, and both a and b are incident to e.

Theorem 3.14 *Given a perfect matching M on a point set $\mathcal{P}$ of size n, it is possible to decide in polynomial time whether this perfect matching can be expanded to a partition of $K(\mathcal{P})$ into plane spanning double stars.*

Proof First, we check whether the perfect matching has any parallel edges. For this we just check for each pair of edges in M whether they are parallel. As the size of M is $\frac{n}{2}$, there are $\binom{\frac{n}{2}}{2} \in \mathcal{O}(n^2)$ pairs so this step takes time

$\mathcal{O}(n^2)$. If any two edges in M are parallel, we know by Theorem 3.7 that M is not expandable.

If M is parallel-free, we define a variable $x_{(e,f)}$ for every pair $\{e, f\}$ of edges in M. We will define a boolean formula F that is satisfiable if and only if M is expandable. For every triple $\{e, f, g\}$ of edges in M we check whether this triple is part of a potential monochromatic crossing. If the triple (e, f, g) is part of a potential monochromatic crossing with both crossing edges incident to e, consider the orientations that make this crossing monochromatic. Define a clause $C = \{l_{(e,f)} \vee l_{(e,g)}\}$, where $l_{(e,f)} = \neg x_{(e,f)}$ if $\{e, f\}$ is left-oriented and $l_{(e,f)} = x_{(e,f)}$ otherwise, and analogously for $l_{(e,g)}$. Define F as the conjunction of all these clauses. Note that F is a 2-CNF with $\mathcal{O}(n^2)$ variables and $\mathcal{O}(n^3)$ clauses. If F has a satisfying assignment $\{x_{(e,f)} = a_{(e,f)}\}_{e,f \in M}$, then we can left-orient every pair of edges $\{e, f\}$ with $a_{(e,f)} = 1$ and right-orient all other pairs in M, and by the construction of F this choice of orientation does not induce any monochromatic crossing. On the other hand, if M is expandable, then there is a choice of orientation for each pair $\{e, f\}$ such that there is no monochromatic crossing, thus setting $x_{(e,f)} = 1$ if and only if the pair $\{e, f\}$ is left-oriented satisfies F. As F is a 2-CNF we can decide whether it is satisfiable in time linear in the number of clauses which gives us a total runtime of $\mathcal{O}(n^3)$. □

Chapter 4

Packings of Plane Spanning Double Stars

In Chapter 3 we have seen that there are point sets BW_k whose complete geometric graphs cannot be partitioned into plane spanning double stars. However, one can check that if k is even, we can still pack $\frac{k}{2} - 1$ plane spanning double stars into $K(BW_k)$. On the other hand, as only one point is not on the boundary of the convex hull, there clearly is a subset of $k - 2$ points whose complete geometric graph can be partitioned into again $\frac{k}{2} - 1$ plane spanning double stars. As it turns out, this is not a coincidence.

Consider a point set $\mathcal{P}$ of size n and a packing of k plane spanning double stars into $K(\mathcal{P})$. Let M be the set of spines of the double stars. We again see that M must be a matching.

Lemma 4.1 *The set of spines M of a packing of k plane spanning double stars into $K(\mathcal{P})$ is a matching.*

Proof Analogous to the proof of Lemma 3.1. □

Corollary 4.2 *Let $\mathcal{P}$ be a point set that allows a packing of k plane spanning double stars into $K(\mathcal{P})$. Then there is a subset $\mathcal{P}'$ of $\mathcal{P}$ of size $2k$ that allows a partition of $K(\mathcal{P}')$ into plane spanning double stars.*

Proof Choose $\mathcal{P}'$ as the set of vertices of the spine matching M. □

On the other hand, we can expand a partition on a subset to a packing on the whole point set.

Lemma 4.3 *Let $\mathcal{P}$ be a point set and let $\mathcal{P}'$ be a subset of $\mathcal{P}$ of size $2k$ that allows a partition of $K(\mathcal{P}')$ into plane spanning double stars. Then $\mathcal{P}$ allows a packing of k plane spanning double stars into $K(\mathcal{P})$.*

For an illustration of the proof see Figure 4.1

Proof Consider an edge e in the spine matching M and a point p in $\mathcal{P} \setminus \mathcal{P}'$. Let E be the plane double star with spine $e = (q, r)$ and let $f = (p, q)$ and $g = (p, r)$ be the edges connecting the point p to the spine e. In order to expand E to a plane spanning double star, we have to add either f or g to E without creating a crossing. Assume for the sake of contradiction that both f and g cross an edge of E. Let s and t be the intersections of f and g with E, respectively. Note that the edge of E that crosses f must be incident to r. Similarly, the edge of E that crosses g is incident to q. As q, r, s and t form a convex quadrilateral, we deduce that E is not plane, which is a contradiction. By induction we can therefore expand E to a plane spanning double star. As the spines form a matching we can do this for every double star in the partition of the subset and the claim follows. □

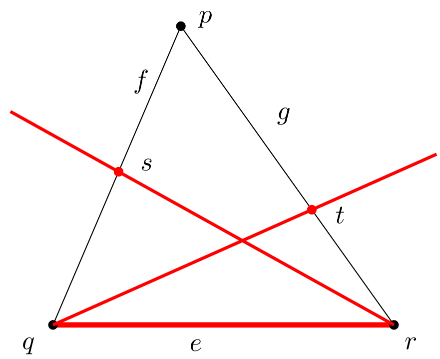


Figure 4.1: Illustration of the proof of Lemma 4.3

Combining Corollary 4.2 and Lemma 4.3 we get the following result:

Theorem 4.4 *Let $\mathcal{P}$ be a point set. Then $K(\mathcal{P})$ allows a packing of k plane spanning double stars if and only if there is a subset $\mathcal{P}'$ of $\mathcal{P}$ of size $2k$ that allows a partition of $K(\mathcal{P}')$ into plane spanning double stars.*

4.1 Packing Plane Spanning Double Stars into Random Point Sets

We define a *random point* p_r by picking a and φ uniformly and independently at random from the intervals $[0, 1]$ and $[0, 2\pi]$ respectively and then setting

$$p_r = (\sqrt{a}\cos(\varphi), \sqrt{a}\sin(\varphi)).$$

A *random point set* $\mathcal{P}_r$ of n points is the union of n independent random points. Note that a random point set is in general position with probability 1.

Combining Theorem 3.12 with Theorem 4.4, we conclude that the existence of k lines with the property that there is at least one point of $\mathcal{P}$ in each unbounded region of the arrangement of the lines, implies that $K(\mathcal{P})$ allows a packing with k plane spanning double stars.

Theorem 4.5 *For a random point set $\mathcal{P}_r$, the expected number of plane spanning double stars that can be packed into $K(\mathcal{P}_r)$ is at least $\lfloor \frac{n}{4} \rfloor$.*

Proof Order the points by their angles φ modulo π. For each point in this order write a 1 if the angle is smaller than π and a 0 otherwise. This induces a random binary string $b_1 b_2 \ldots b_n$ with $\mathbb{P}[b_i = 1] = \mathbb{P}[b_i = 0] = \frac{1}{2}$. Consider an alternating substring b of even length $2k$ and assume without loss of generality that it starts with a 0. Let $\varphi_1, \ldots, \varphi_k$ be the angles of the points contributing a 1 to the substring. For each $i \in \{1, \ldots, k\}$ define a line ℓ_i through the origin such that the angle between ℓ_i and the x-axis is $\varphi_i + \varepsilon$ for some small $\varepsilon > 0$ (see Figure 4.2). Then there is a point contributing to the substring b in every unbounded region of the arrangement defined by the lines $\ell_1, \ldots, \ell_k$, and by Theorem 3.12, the complete geometric graph on these points can be partitioned into plane spanning double stars. Therefore, by Theorem 4.4, $K(\mathcal{P}_r)$ allows a packing with k plane spanning double stars. As in a random binary string the expected length of a longest alternating substring is $\lfloor \frac{n}{2} \rfloor$, the theorem follows. $\square$

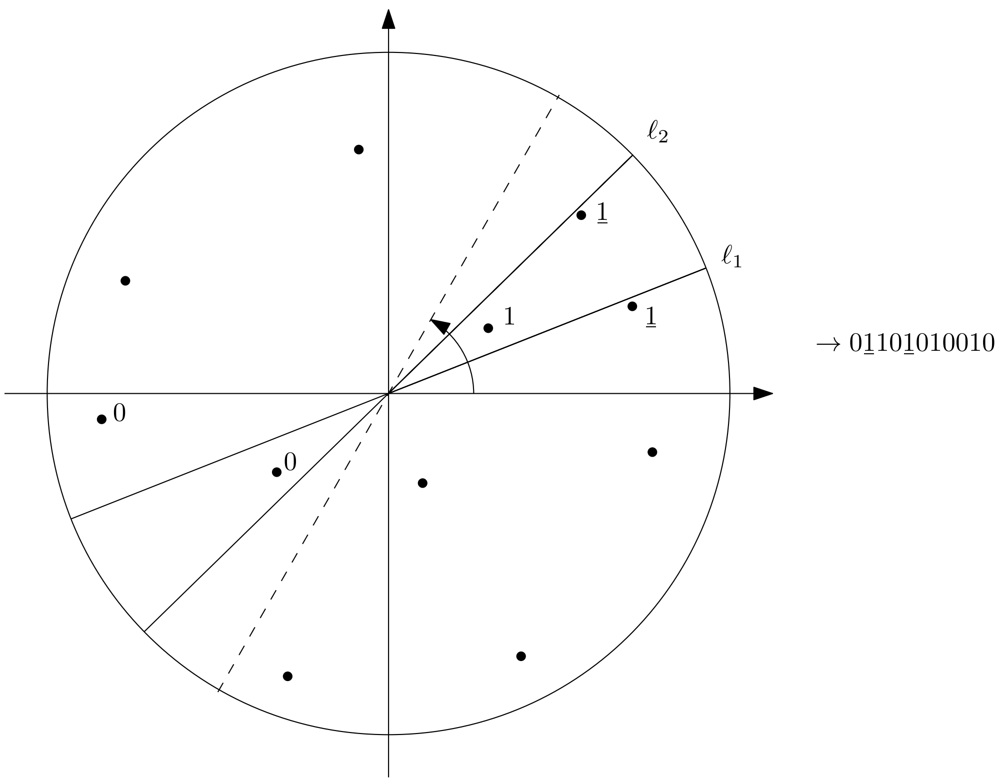


Figure 4.2: Illustration of the proof of Theorem 4.5

Note that the lines that we constructed in the proof are concurrent. For general point sets, this restriction is rather strong and not necessary for our purposes.

4.2 Point Sets that only allow small Packings with Plane Spanning Double Stars

In Chapter 3 we constructed point sets BW_k whose complete geometric graphs only allow packings with $\frac{k}{2} - 1$ plane spanning double stars. But there are point sets where the largest packings with plane spanning double stars are even smaller.

For any $m \in \mathbb{N}$, define a point set $\mathcal{R}_m$ as follows: Place $9m$ points in convex position, partitioned into three sets A_i, $i \in \{1, 2, 3\}$, of $3m$ consecutive points each. Then place a set B of m points in the interior, such that for any $b \in B$ and any union $A_i \cup A_j$, b lies outside of the convex hull of $A_i \cup A_j$. See Figure 4.3 for an illustration.

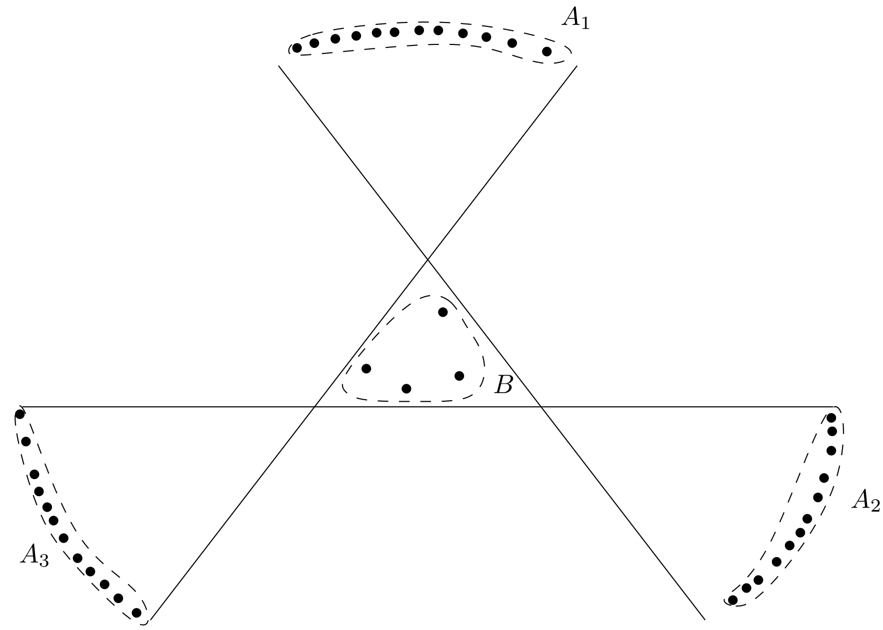


Figure 4.3: A point set whose complete geometric graph only allows a packing with $\frac{9}{20}n$ plane spanning double stars

We will show that any expandable matching on $\mathcal{R}_{2k}$ can have at most $9k$ edges. As $\mathcal{R}_{2k}$ has $n = 20k$ points, this implies that at most $\frac{9}{20}n$ plane spanning double stars can be packed into $K(\mathcal{R}_{2k})$. But we first need to prove two auxiliary lemmas.

Lemma 4.6 *Let M be a perfect matching on $\mathcal{R}_m$ that has no edge between two points in B. Then we need to take away at least m edges from M in order to be left with an expandable matching.*

Proof We prove this by induction on m. For $m = 1$ the point set $\mathcal{R}_1$ is exactly the point set BW_{10}.

For the inductive step, assume that $m \geq 2$. Let A_i', $i \in \{1,2,3\}$, be the subset of points of A_i that are not matched with a point in B. Note that $2m \leq |A_i'| \leq 3m$ and $5m \leq |A_i' \cup A_j'| \leq 6m$, $i \neq j$, and $|\bigcup_{i=1}^{3} A_i'| = 8m$. Let E_1 be the set of edges in M between points in A_1' and let F_1 be the set of edges in M between a point in A_2' and a point in A_3'. Define E_2, E_3, F_2 and F_3 analogously. As $|\bigcup_{i=1}^{3} A_i'| = 8m$, we have that $|\bigcup_{i=1}^{3} E_i \cup \bigcup_{j=1}^{3} F_j| = 4m$. Also, as $|A_i'| \leq 3m$, for i, j and k all different we have $|E_i \cup F_j \cup F_k| \leq 3m$ and in particular $|F_j \cup F_k| \leq 3m$.

Note that any edges $e \in E_i$ and $f \in F_i$ are parallel. The same holds for any edges $e \in E_i$ and $f \in E_j$, where $i \neq j$. Also, any edge between B and A_i is parallel to any edge in E_j for $i \neq j$. Recall that by Theorem 3.7 no two edges in an expandable matching can be parallel.

First assume that there is an i such that there is an edge e between B and A_i, $|F_i| \geq 2$ and $|F_j| \geq 1$ for $j \neq i$. We will call this assumption the *subset assumption*. Without loss of generality let $i = 1$. Let M' consist of the edge e, as well as two edges in F_1, one edge of F_2 and one edge of F_3. Then the vertex set of M' is $\mathcal{R}_1$ and the vertex set of $M \setminus M'$ is $\mathcal{R}_{m-1}$. Thus, by the inductive hypothesis, we need to take away at least one edge from M' and at least $m - 1$ edges from $M \setminus M'$ in order to get an expandable matching.

We will now consider the cases where the subset assumption is not satisfied. First, assume that without loss of generality $|F_1| = 0$. As $|A_2' \cup A_3'| \geq 5m$ and $|F_2 \cup F_3| \leq 3m$, we deduce that $|E_2 \cup E_3| \geq m$. Note that every edge in E_2 is parallel to any edge in $E_3 \cup E_1 \cup F_2$ and every edge in E_3 is parallel to any edge in $E_2 \cup E_1 \cup F_3$. Thus, if we want to leave any edge in E_2, we have to take away all edges in $E_3 \cup E_1 \cup F_2$. But as $|E_2 \cup F_3| = |E_2 \cup F_3 \cup F_1| \leq 3m$, we have $|E_3 \cup E_1 \cup F_2| \geq m$. The same argument applies if we want to leave any edge in E_3. The only other thing that we can do is to take away the edges in $E_2 \cup E_3$, and we have seen before that $|E_2 \cup E_3| \geq m$. Thus if $|F_i| = 0$ for some i, we also need to take away at least m edges in order to get an expandable matching.

So, from now on we can assume that none of the F_i's is empty. Suppose now that $|F_1| = |F_2| = |F_3| = 1$. Then $|E_i| \geq m - 1$ for $i \in \{1,2,3\}$. As any edge in E_i is parallel to any edge in E_j for $i \neq j$, we have to take away all edges of $E_i \cup E_j$ for some $i, j \in \{1,2,3\}$ with $i \neq j$. As each E_i has size $|E_i| \geq m - 1$, we deduce that in this case we need to take away at least $2m - 2 \geq m$ edges in order to get an expandable matching.

This means that we can also assume that there are at least 2 edges in some F_i. Hence if there is an edge between B and A_j for every j, then the subset assumption is satisfied. So, assume without loss of generality that there is no edge between B and A_1. Then there either are edges between B and A_2 and B and A_3, or without loss of generality all edges incident to a point in B are between B and A_3.

First assume that there is no edge between B and A_1, i.e. $|A_1'| = 3m$, but there are edges between B and A_2 and B and A_3. If $|F_2| \geq 2$ or $|F_3| \geq 2$, then the subset assumption is satisfied, so assume that $|F_2| = |F_3| = 1$. This, together with $|A_1'| = 3m$, implies that $|E_1| = \frac{3m-2}{2} \geq m$. Let G be the set of edges incident to a point in B. Note that $|G| = m$. As there is no edge between B and A_1, any edge in G is parallel to any edge in E_1. Thus we either have to take away every edge in G or every edge in E_1. So again we need to take away at least m edges in order to get an expandable matching.

Finally, assume that all edges incident to a point in B are without loss of generality between B and A_3, i.e. $|A_3'| = 2m$. If $|F_3| \geq 2$, then the subset assumption is satisfied, so assume $|F_3| = 1$. As $|A_3'| = 2m$, we deduce that $|E_3 \cup F_1 \cup F_2| \leq 2m$ and thus $|F_3 \cup E_1 \cup E_2| \geq 2m$. As $|F_3| = 1$ we thus have that $|E_1 \cup E_2| \geq 2m - 1 \geq m$. Let again G be the set of edges incident to a point in B. As all edges in G are between B and A_3, any edge in G is parallel to any edge in $E_1 \cup E_2$, so again we need to take away either all of G or all of $E_1 \cup E_2$. As both sets contain at least m edges, we again need to take away at least m edges in order to get an expandable matching. This concludes the inductive step. $\square$

Lemma 4.7 *Let $k \in \mathbb{N}$ and let V_1, V_2, V_3 be pairwise disjoint point sets with $4k + 2 \leq |V_i| \leq 6k$ for every $i \in \{1, 2, 3\}$. Let N be a perfect matching on $V = V_1 \cup V_2 \cup V_3$. Then there exists a sub-matching $N' \subset N$ such that N' has exactly 6 vertices in each V_i.*

Proof First note that for $k = 1$ we have $|V_i| = 6$ for every i and we can thus choose $N' = N$.

Assume for the sake of contradiction that there is a counterexample to the claim for some $k \geq 2$ and consider the smallest k for which such a counterexample exists. For this k, let V_1, V_2, V_3 be the three sets in a counterexample with the smallest number of points and let N be the corresponding matching. At least one V_i must have size $|V_i| > 6k - 6$, as else N would already be a counterexample for $k - 1$. Assume without loss of generality that $|V_1| = 6k - m$ for some $0 \leq m \leq 5$. Note that there can be no edge in N between two points in V_1, as else we could remove this edge and get a smaller counterexample. Similarly, we must have $|V_2| = |V_3| = 4k + 2$, as else we could take away an edge between V_1 and V_2 or V_3 and get a smaller counterexample.

We claim that we can take away $6 - m$ edges from N that are incident to a point in V_1 and get a counterexample for $k - 1$, which would be a contradiction. As we noted before, there are no edges between two points in V_1, so there are $6k - m$ edges that we could remove. However, we need to be careful that both V_2 and V_3 have size at least $4(k-1) + 2 = 4k - 2$. For $m \geq 2$, this is not a problem, as we do not need to remove more than four edges. For $m = 0$ or $m = 1$ we note that there must be at least $6k - m - (4k + 2) = 6k - m - 2 \geq 2 - m$ edges between V_1 and V_2, and analogously for V_1 and V_3. Hence we can without loss of generality remove 4 edges between V_1 and V_2 and $2 - m$ edges between V_1 and V_3, and we get a counterexample for $k - 1$, which is a contradiction to our choice of k. □

We are now ready to prove the main result of this section.

Theorem 4.8 *There are point sets $\mathcal{P}$ of size $n = 20k$ with the property that at most $\frac{9}{20}n$ plane spanning double stars can be packed into $K(\mathcal{P})$.*

Proof For any k, consider the point set $\mathcal{R}_{2k}$ and note that it indeed has $20k$ points. Let M be any perfect matching on $\mathcal{R}_{2k}$. We claim that we need to take away at least k edges from M in order to be left with an expandable matching. We prove this claim by induction on k.

For $k = 1$, consider a perfect matching M on the point set $\mathcal{R}_2$. If there is no edge in M between the two points in B, then Lemma 4.6 implies that we need to take away at least two edges from M to get an expandable matching. So, assume that there is an edge between the two points in B. If two edges in M are parallel, we need to take away at least one edge of M to get an expandable matching. If no edges in M are parallel, then without loss of generality there are two edges e and f between A_1 and A_2 that cross. Then e and f form a cross-blocker together with the edge between the two points in B. So also in this case we need to take away at least one edge of M to get an expandable matching, which concludes the base case.

For the inductive step, assume that the claim is true for $k - 1$. Consider a perfect matching M on the point set $\mathcal{R}_{2k}$. If there is no edge in M between two points in B, then Lemma 4.6 implies that we need to take away at least $2k$ edges from M to get an expandable matching. So, assume that there is an edge e between two points in B. By Lemma 4.7, there is a subset $M' \subset M$ of the perfect matching M such that the vertex set of $M' \cup \{e\}$ is $\mathcal{R}_2$. Then the vertex set of $M \setminus (M' \cup \{e\})$ is $\mathcal{R}_{2(k-1)}$. By the inductive hypothesis we need to take away at least $k - 1$ edges from $M \setminus (M' \cup \{e\})$ and at least one edge from $M' \cup \{e\}$. So in total, we need to take away at least k edges from M to get an expandable matching, which concludes the inductive step.

So, any expandable matching on $\mathcal{R}_{2k}$ has at most $10k - k = 9k$ edges and thus at most $9k = \frac{9}{20}n$ plane spanning double stars can be packed into $K(\mathcal{R}_{2k})$. □

Chapter 5

Packing Plane Double Stars into special Point Sets

In this chapter we will use the results on expandable matchings of the previous chapters to construct large expandable matchings on two special types of point sets: Horton point sets and point sets with many halving lines.

5.1 Horton Point Sets

Horton Point Sets [11] are sets of points with no empty convex 7-gon. They are constructed as follows:

Definition 5.1 (Horton Point Set) *For any k, let $a_1 a_2 \dots a_k$ be the binary representation of the integer i, $0 \leq i < 2^k$, including leading 0's. Define $c = 2^k + 1$ and $d(i) = \sum_{j=1}^{k} a_j c^{j-1}$. Let p_i be the point $(i, d(i))$. The Horton Point Set $\mathcal{S}_k$ is the set of points $\{p_i \mid 0 \leq i < 2^k\}$*

See Figure 5.1 for a (scaled) picture of $\mathcal{S}_4$.

The Horton Point Sets $\mathcal{S}_k$ have some nice properties, as proved in [11]. We will use that

1. $L = \{p_i \mid i < 2^{k-1}\}$ is the left half of $\mathcal{S}_k$,
2. $R = \{p_i \mid i \geq 2^{k-1}\}$ is the right half of $\mathcal{S}_k$,
3. $B = \{p_i \mid i \text{ is even}\}$ is the bottom half of $\mathcal{S}_k$,
4. $T = \{p_i \mid i \text{ is odd}\}$ is the top half of $\mathcal{S}_k$,
5. all points of T are above any line through two points in B, and
6. all points of B are below any line through two points in T.

Theorem 5.2 *For any integer k, $K(\mathcal{S}_k)$ can be partitioned into plane spanning double stars.*

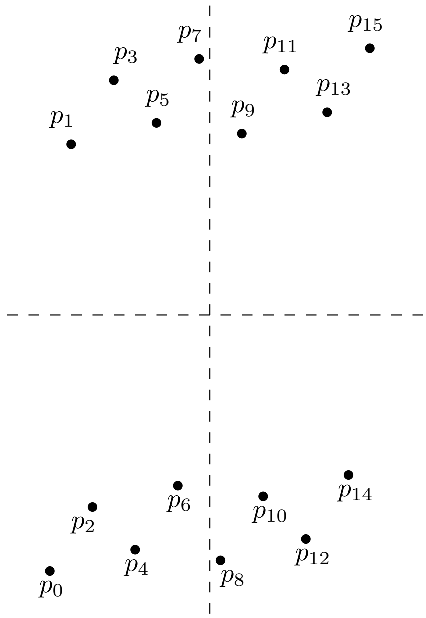


Figure 5.1: A qualitative depiction of the Horton Point Set $\mathcal{S}_4$

Proof For every i, $0 \leq i < 2^{k-1}$, match p_i with p_{2^k-1-i}. We claim that any two edges in this perfect matching M cross.

We start by making two easy observations. If i is even then $2^k - 1 - i$ is odd, whereas if i is odd then $2^k - 1 - i$ is even, implying that for any edge in M one endpoint is in B and the other one is in T. Also, any subset of four points consisting of two points b_1 and b_2 in B and two points t_1 and t_2 in T form a convex quadrilateral as both t_1 and t_2 lie above the line through b_1 and b_2 and both b_1 and b_2 lie below the line through t_1 and t_2, which means that the segments (b_1, b_2) and (t_1, t_2) are parallel.

Consider the edges e_i and e_j in M with respective left endpoints p_i and p_j, where $i < j$. Then p_i lies to the left of p_j, while p_{2^k-1-i} lies to the right of p_{2^k-1-j}. If both i and j are even, then p_i and p_j are in B, while p_{2^k-1-i} and p_{2^k-1-j} are in T. Since p_i lies to the left of p_j but p_{2^k-1-i} lies to the right of p_{2^k-1-j}, and the four point form a convex quadrilateral, the two edges e_i and e_j must cross. The same argument holds if both i and j are odd. If on the other hand i is even and j is odd, or i is odd and j is even, then without loss of generality p_i is in B and p_j is in T. Thus p_{2^k-1-i} is in T and p_{2^k-1-j} is in B. Also, p_i and p_j are in L, while p_{2^k-1-i} and p_{2^k-1-j} are in R. Hence both edges e_i and e_j cross any line separating L and R, as well as any line separating B and T. Together with the fact that the points p_i, p_j, p_{2^k-1-i} and p_{2^k-1-j} form a convex quadrilateral, this again implies that the two edges cross.

We deduce that any two edges in M cross, i.e. M is a crossing family. Hence M satisfies the conditions of Theorem 3.10 and can thus be expanded to a partition of $K(\mathcal{S}_k)$ into plane spanning double stars. $\square$

5.2 Point Sets with many Halving Lines

We now consider point sets with many halving lines as constructed by Edelsbrunner and Welzl [8], which we will call *EW-sets*. For n points, these point sets have $\Omega(n \log n)$ halving lines. They are constructed inductively.

First, consider three rays emanating from the origin, with any two rays enclosing an angle of $\frac{2\pi}{3}$. Place two points on every ray such that no two points coincide and no point coincides with the origin. This point set is $\mathcal{Q}_1$ and is depicted in Figure 5.2. Given $\mathcal{Q}_{k-1}$, we now construct $\mathcal{Q}_k$. Consider again three rays emanating from the origin, with any two rays enclosing an angle of $\frac{2\pi}{3}$. For each ray, draw a small wedge with angle ε containing the ray. Place a copy of $\mathcal{Q}_{k-1}$ in each wedge, flattened by an affine transformation in such a way that any line through two points in one of the copies of $\mathcal{Q}_{k-1}$ separates the other two copies. This new point set is $\mathcal{Q}_k$. Note that $\mathcal{Q}_k$ has $2 \cdot 3^k$ points.

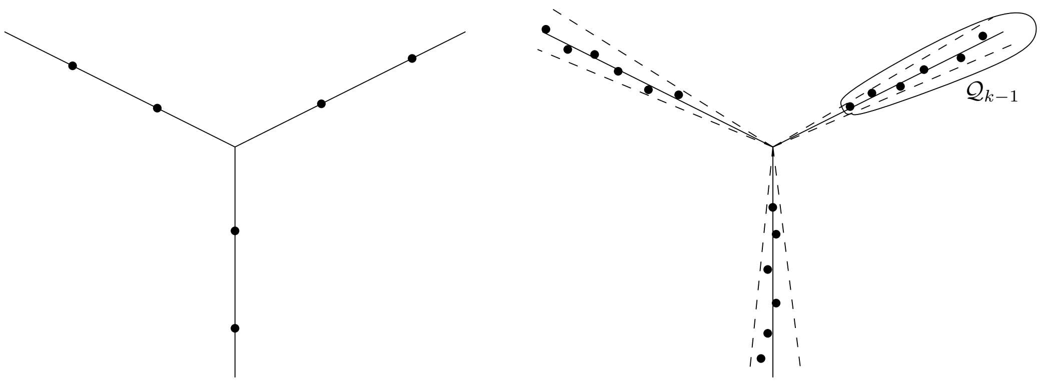


Figure 5.2: The point sets $\mathcal{Q}_1$ (left) and $\mathcal{Q}_k$ (right)

We will again construct an expandable matching on these point sets. For this we first need to name a few configurations. For a drawing of these configurations, see Figure 5.3.

A *Y-configuration* is a triple $Y = \{e, f, g\}$ of three pairwise non-incident edges such that e and f cross, g stabs both e and f, g does not intersect the convex hull of e and f, and both vertices of g see every vertex of e and f in Y.

An *A-configuration* is a triple $A = \{e, f, g\}$ of three pairwise non-incident edges such that f stabs e, and g stabs e and f with different stabbing vertices.

Note that in an A-configuration, the edge g is completely contained in the convex hull of the edges e and f, as the intersections of the supporting line ℓ_g of g with e and f lie on different sides of g.

A *T-configuration* is a triple $T = \{e, f, g\}$ of three pairwise non-incident edges such that e and f cross, and both e and f stab g.

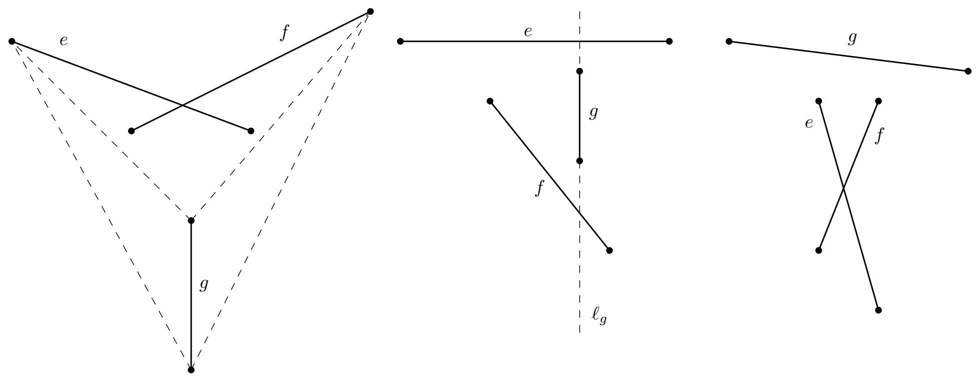


Figure 5.3: A Y-configuration (left), an A-configuration (middle) and a T-configuration (right)

Lemma 5.3 *Let M be a matching of size at least 3 with the property that every subset of three edges of M is either a crossing family of size 3, a Y-configuration, an A-configuration or a T-configuration. Then left-orienting each pair of edges induces a partition of the complete geometric graph of the vertices of M into plane spanning double stars.*

Proof Note that M is parallel-free and thus "left-orienting" is well defined for every pair of edges in M. Consider the partition of the complete geometric graph of the vertices of M given by left-orienting each pair in M and remember that M is now the spine matching. Assume for the sake of contradiction that one of the spanning double stars in the partition is not plane. Then the two crossing edges are incident to at most three of the spines in M. Thus there is a subset $M' \subset M$ of size 3 with the property that left-orienting each pair in M' induces a monochromatic crossing.

However, we claim that for every possible M', left-orienting each pair of M' induces a partition of the complete geometric graph of the vertices of M' into plane spanning double stars. For crossing families, A-configurations and T-configurations the claim follows from Lemma 3.11. For Y-configurations see Figure 5.4. This gives us a contradiction and we deduce that all spanning double stars in the partition are indeed plane. $\square$

Lemma 5.4 *Let $\mathcal{C}$ be the set of two copies of $\mathcal{Q}_{k-1}$ in $\mathcal{Q}_k$. Then $\mathcal{C}$ has a crossing-family of size $|\mathcal{Q}_{k-1}|$.*

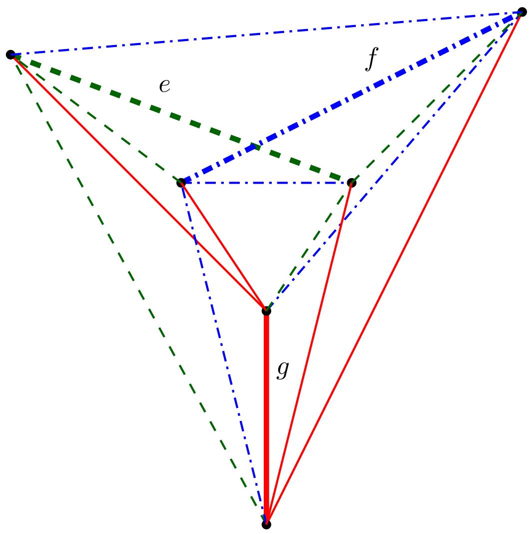


Figure 5.4: The expansion of a Y-configuration determined by left-orienting each pair of edges

Proof Let A and B be the two copies of $\mathcal{Q}_{k-1}$ and assume without loss of generality that the rays defining the wedges in which A and B lie both point upwards. Order the points in A and B by their y-coordinate. Iteratively match the lowest unmatched point in A with the highest unmatched point in B to get a matching M. Let $e_0 = (a_0, b_0)$ and $e_1 = (a_1, b_1)$ be two edges in M with $a_0, a_1 \in A$, $b_0, b_1 \in B$ and a_0 above a_1. Then b_0 is below b_1. By the construction of the point set, the line through a_0 and a_1 separates B and the third copy of $\mathcal{Q}_{k-1}$, while the line through b_0 and b_1 separates A and the third copy of $\mathcal{Q}_{k-1}$. This implies that (a_0, a_1) and (b_0, b_1) are parallel, hence the four points a_0, a_1, b_0 and b_1 form a convex quadrilateral. As a_0 is above a_1 but b_0 is below b_1, we deduce that the two edges e_0 and e_1 cross. As this holds for any two edges in M, the claim follows. $\square$

We are now ready to construct an expandable matching on $\mathcal{Q}_k$, see Figure 5.5 for an example of such a matching on $\mathcal{Q}_3$.

Let A, B and C be the three copies of $\mathcal{Q}_{k-1}$, with A lying in the wedge W_A, B lying in the wedge W_B and C lying in the wedge W_C. Assume without loss of generality that the rays defining W_A and W_B both point upwards. Then the ray defining W_C points downwards. Let M_1 be the crossing family defined by $A \cup B$ as constructed in Lemma 5.4. Let C_1, C_2, C_3 be the three copies of $\mathcal{Q}_{k-2}$ in C, ordered such that the lowest point of C_i lies below the lowest point of C_j if $i < j$. Define M_2 as the crossing family defined by $C_1 \cup C_2$ and let M_3 be the largest crossing family in C_3. Set $M_Q = M_1 \cup M_2 \cup M_3$ and note that $|M_Q| \geq \frac{n}{3} + \frac{n}{9} + \frac{n}{27} = \frac{13}{27}n$, where $|\mathcal{Q}_k| = n$.

Theorem 5.5 *Let $\mathcal{Q}_k$ be an EW-set, with $|\mathcal{Q}_k| = n$. Then at least $\frac{13}{27}n$ plane spanning double stars can be packed into $K(\mathcal{Q}_k)$.*

Proof Consider the matching M_Q, and remember that $|M_Q| \geq \frac{13}{27}n$. We will show that M_Q is expandable. The result then follows from Theorem 4.4.

We claim that any three edges of M_Q either form a crossing family of size 3, a Y-configuration, an A-configuration or a T-configuration. Pick any three edges in M_Q. If they all lie in the same M_i, then they form a crossing family. If two of them lie in M_i and one in M_j, with $i < j$, then they form a Y-configuration. If one of them lies in M_i and two in M_j, again $i < j$, then they form a T-configuration. If they all lie in different M_i's, then they form an A-configuration. It thus follows from Lemma 5.3 that M_Q is expandable. □

It might be possible to add even more edges to the matching M_Q without losing expandability, but we would have to be very careful as we could get configurations where left-orienting each pair might induce monochromatic crossings.

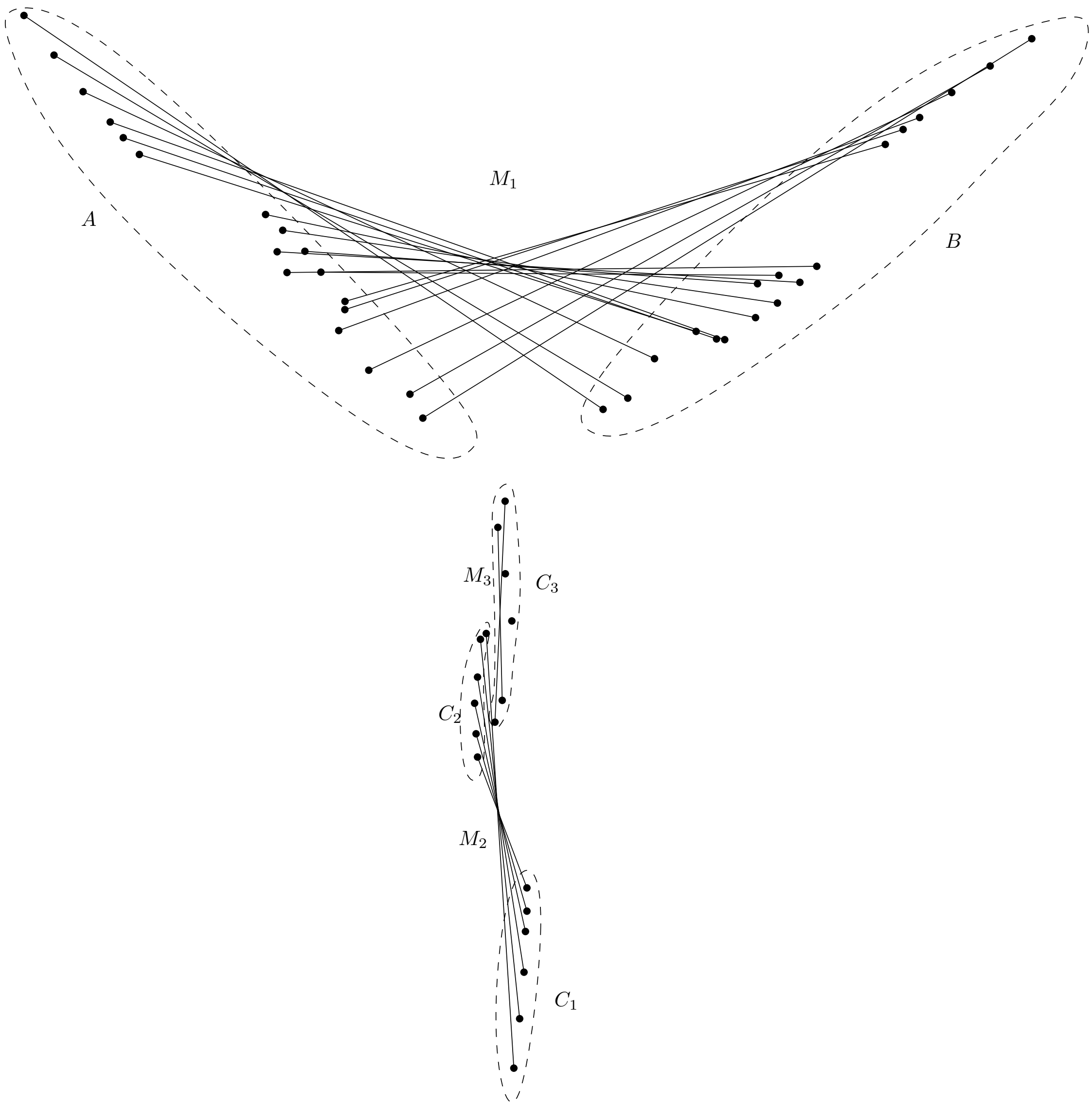


Figure 5.5: An expandable matching on a qualitative drawing of $\mathcal{Q}_3$

Part III

Miscellaneous

Chapter 6

Partitions into Plane Spanning Paths

In this chapter we consider another special case of trees, namely paths. It is known that there are point sets whose complete geometric graph cannot be partitioned into plane spanning paths [2]. In fact, there are point sets that only allow rather small packings with plane spanning paths:

Remark 6.1 *There are point sets of size n, where n is even, whose complete geometric graph does not allow a packing of more than $\lceil \frac{n}{3} \rceil$ plane spanning paths.*

Proof Each spanning path has $n-1$ edges, which is an odd number as n is even. Thus each spanning path contains a perfect matching, and therefore a packing of k plane spanning paths induces a packing of k plane perfect matchings. Biniaz et al. [5] have shown that there are point sets whose complete geometric graph does not allow a packing of more than $\lceil \frac{n}{3} \rceil$ plane perfect matchings. □

On the other hand, for an even number of points in convex position, the complete geometric graph can be partitioned into plane spanning paths [6]. The same holds for point sets that are crossing-dominated by convex position: if a point set $\mathcal{Q}$ is crossing-dominated by a point set $\mathcal{P}$ in convex position, by definition of crossing-dominance there is a bijection from $\mathcal{Q}$ to $\mathcal{P}$ such that any two edges that cross in $K(\mathcal{Q})$ also cross in $K(\mathcal{P})$. Thus any partition of $K(\mathcal{P})$ into plane spanning paths can be mapped back to $K(\mathcal{Q})$ and all the paths in the partition stay plane under this mapping.

For any point set $\mathcal{P}$, we denote the set of points that lie on the boundary of the convex hull of $\mathcal{P}$ by $Ext(\mathcal{P})$, and we call a point in $Ext(\mathcal{P})$ an *extreme point*. We will prove the following result:

Theorem 6.2 *Let $\mathcal{W}$ be a point set with $|Ext(\mathcal{W})| = |\mathcal{W}| - 1$. Then $K(\mathcal{W})$ allows a partition into plane spanning paths if and only if $\mathcal{W}$ is crossing-dominated by convex position.*

In order to prove this we try to color the edges of the induced geometric graph of such a point set in a way that each color class is a plane spanning path. We will then see that the coloring must have a very specific structure and that the one point that is not in the convex hull can only lie in very few places. We first start with a few definitions and lemmas.

Let $\mathcal{W}$ be a point set with $|Ext(\mathcal{W})| = |\mathcal{W}| - 1$ and $|\mathcal{W}| = 2n + 2$ (this value is chosen for ease of notation). Denote the points in $Ext(\mathcal{W})$ as $v_0, v_1, \ldots, v_{2n}$, enumerated in counter-clockwise order, and let v_{int} be the point not in $Ext(\mathcal{W})$.

We are given $n+1$ colors $c[0], \ldots, c[n]$, and we want to assign these colors to the edges of $K(\mathcal{W})$ in such a way that every edge gets exactly one color and each color class is a spanning path. For the rest of this chapter, we will consider every partition of $K(\mathcal{W})$ into spanning paths to be such a coloring, and we will say that two edges have the same color if they are in the same path of the partition.

For every edge $e = (p, q)$ with $p, q \in Ext(\mathcal{W})$ let the *span of e*, denoted by $span(e)$, be the number of edges on a shortest path between p and q that is contained in the boundary of the convex hull, i.e. $span((v_i, v_j)) = \min\{j - i, i + (2n - j) + 1\}$ for $i < j$. Let E be the set of edges e between vertices in $Ext(\mathcal{W})$ such that $span(e)$ is maximal, i.e. $span(e) = n$. Note that each point p in $Ext(\mathcal{W})$ is incident to exactly two edges of E and that E defines a "star-shaped "cycle of length $2n+1$ in $K(\mathcal{W})$.

For two arbitrary edges $e = (p, q)$ and $f = (p, s)$ between points in $Ext(\mathcal{W})$ that are incident to a common point p, we call the pair $\{e, f\}$ a *wedge* and we say that p is the *apex* of the wedge $\{e, f\}$. We say that a wedge is *monochromatic* if both edges have the same color. Consider the shortest path between q and s and let V be the set of vertices in this path. A point s lies *inside the wedge* $\{e, f\}$ if it lies in the convex hull of $V \cup \{p, q, r\}$.

Lemma 6.3 *In every partition of $K(\mathcal{W})$ into plane spanning paths, the edges in E are partitioned into n wedges and one single edge e_0. Also, the apexes of these wedges all lie on the same side of the supporting line ℓ of e_0 and all vertices in $Ext(\mathcal{W})$ on this side are apexes of such a wedge.*

We will call these wedges the *main wedges*.

Proof If $n = 1$, then E is a triangle. As we only have two colors in this case, the claim follows immediately. So assume $n > 1$.

Every non-incident pair of edges in E crosses, thus any two edges in E of the same color must form a wedge, and there can be at most two edges in E with the same color. As $|E| = 2n + 1$, we conclude that E must be partitioned into n wedges and one single edge e_0.

To see that all apexes lie on the same side of ℓ, we first note that if two points are connected by an edge in E, at most one of them can be an apex of a wedge, as else there would be three edges of the same color. Also, there must be n points in $Ext(\mathcal{W})$ that are apexes, and none of them is incident to e_0. It follows that every edge except e_0 must be incident to an apex. Thus the vertices along the cycle defined by E alternate between being an apex and not being an apex, except for the two vertices that are incident to e_0, and as each edge that is not incident to e_0 crosses e_0, this concludes the proof. □

Suppose now that we have colored E as in Lemma 6.3. Assume without loss of generality that ℓ is vertical and that the apexes of the main wedges lie to the left of ℓ. Let v_0 be the upper vertex of e_0, i.e. $e_0 = (v_0, v_{n+1})$. Consider two arbitrary edges e and f between points in $Ext(\mathcal{W})$, each with one vertex left of ℓ and one vertex right of ℓ and assume that the left vertex of e lies above the right vertex of f. Then e and f cross if and only if the right vertex e lies below the right vertex of f. In other words, for $e = (v_a, v_{n+b})$ and $f = (v_i, v_{n+j})$ with $0 < a < i < n+1$ and $0 < b, j < n+1$, e and f cross if and only if $b < j$.

Let the main wedge at the vertex v_i for $1 \leq i \leq n$ have color $c[i]$, i.e. the main wedge of color $c[i]$ consists of the edges (v_i, v_{n+i}) and (v_i, v_{n+i+1}). Let the color of e_0 be $c[0]$. Note that every vertex v_{n+i}, $1 \leq i \leq n$, is incident to edges with color $c[i]$ and $c[i-1]$. See Figure 6.1 for an illustration.

The following lemma holds for any point set, so we state it in more general terms.

Lemma 6.4 (Wedge Lemma) *Let $\mathcal{P}$ be a point set and let p, q and r be points in $Ext(\mathcal{P})$. Consider the edges $e = (p, q)$ and $f = (p, r)$ and let W be the set of points in $\mathbb{R}^2$ that lie inside the wedge $\{e, f\}$. Assume that $span(e) \geq span(f) > 1$. If there exists a point $s \in \mathcal{P}$ that lies in W, then $\mathcal{P}$ does not allow a partition of $K(\mathcal{P})$ into plane spanning paths where e and f have the same color.*

Proof As p, q and r are in $Ext(\mathcal{P})$, the edges e and f divide $Conv(\mathcal{P})$ into three parts A, W and B, with e lying on the boundary of A and f lying on the boundary of B. See Figure 6.2 for an illustration. As $span(e) \geq span(f) > 1$, there are points of $\mathcal{P} \setminus \{p, q, r\}$ in both A and B. Let $a \in A$ and $b \in B$ be such points. Assume for the sake of contradiction that there exists a partition of $K(\mathcal{P})$ into plane spanning paths where both e and f are colored red. Then the red path must also go through a and b, and as the path is plane, the sub-path from a to b must visit a, q, p, r, and b in this order. On the other hand, any point in the interior of W can only be connected to p, q or r with a red edge, as there would be a red crossing otherwise. As there is at least one point s inside W this implies that in the red subgraph of the partition, p, q or r has degree at least 3, so the red subgraph cannot be a path, which is a contradiction. □

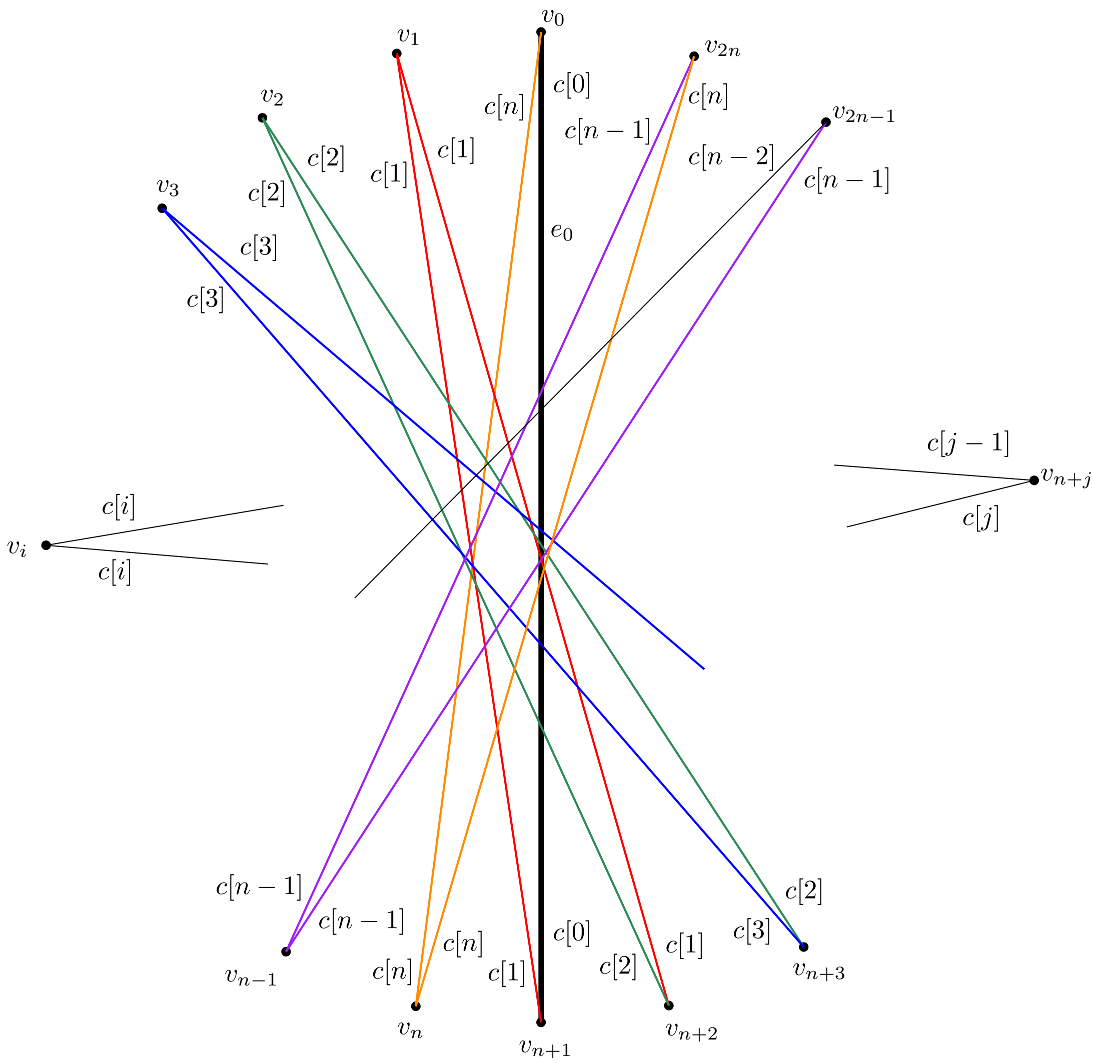


Figure 6.1: The coloring of the edges with maximal span

For any subset R of $\mathbb{R}^2$, we say that R is *covered* by a wedge $\{e, f\}$ if every point in R lies inside the wedge $\{e, f\}$. The next lemma will show, that in any partition of $K(\mathcal{W})$ into plane spanning paths, a large part of $Conv(\mathcal{W})$ is covered by monochromatic wedges.

Any plane path with the property that every edge is between points in $Ext(\mathcal{W})$ and has span larger than 1, and that for any three consecutive points p, q, s on this path, the span of the edge (p, s) (that is not part of the path) is exactly 1, is called a *zig-zag path*. Note that any two consecutive edges of a zig-zag path form a wedge. We say that a zig-zag path Z covers a subset R of $\mathbb{R}^2$ if every point in R lies inside a wedge defined by two consecutive edges in Z.

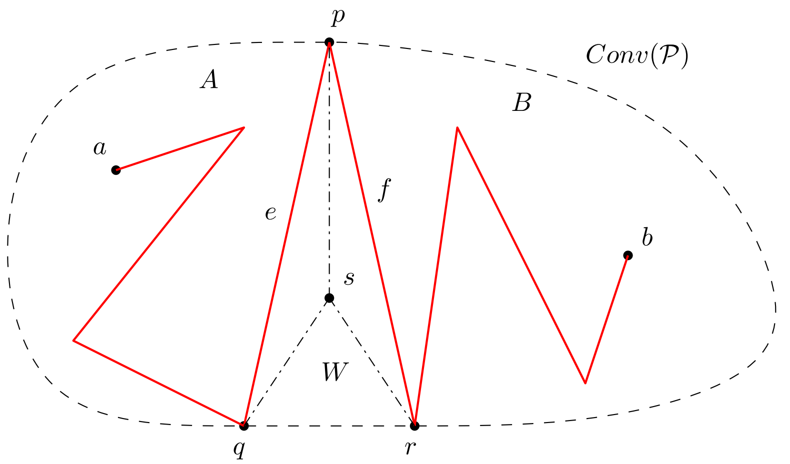


Figure 6.2: An Illustration of the proof of the Wedge Lemma

Let a be the intersection of (v_0, v_2) and (v_1, v_{2n}) and let b be the intersection of (v_{n-1}, v_{n+1}) and (v_n, v_{n+2}). Let T_a be the triangle defined by v_0, v_1 and a and let T_b be the triangle defined by v_n, v_{n+1} and b. See Figure 6.3 for an illustration.

If n is even, we call the colors $c[\frac{n}{2}]$ and $c[\frac{n}{2}+1]$ the *central colors*. If n is odd, we call the colors $c[\frac{n-1}{2}]$, $c[\frac{n+1}{2}]$ and $c[\frac{n+1}{2}+1]$ the *central colors*.

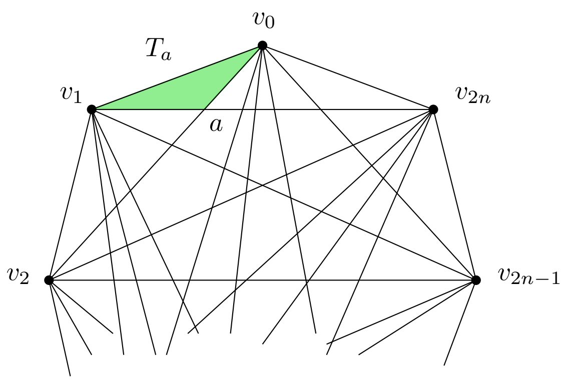


Figure 6.3: The triangle T_a

Lemma 6.5 (Zig-zag Lemma) *Let $Ext(\mathcal{W}) = \{v_0, v_1, \ldots, v_{2n}\}$ be enumerated in counter-clockwise order with the edges of maximal span colored as in Lemma 6.3. Then in every partition of $K(\mathcal{W})$ into plane spanning paths, there is a monochromatic zig-zag path for every central color, and these zig-zag paths cover all of $Conv(\mathcal{W})$ except T_a and T_b.*

The proof of this lemma is rather long and technical, so we postpone it to the next section.

With the Wedge Lemma and the Zig-zag Lemma we are able to prove Theorem 6.2

Proof (of Theorem 6.2) Let $\mathcal{W}$ be a point set with $|Ext(\mathcal{W})| = |\mathcal{W}| - 1$, with $Ext(\mathcal{W}) = \{v_0, v_1, \ldots, v_{2n}\}$ enumerated in counter-clockwise order. We want to show that $K(\mathcal{W})$ allows a partition into plane spanning paths if and only if $\mathcal{W}$ is crossing-dominated by convex position.

If $\mathcal{W}$ is crossing-dominated by convex position, then it is clear that $K(\mathcal{W})$ allows a partition into plane spanning paths.

For the other direction, assume that $K(\mathcal{W})$ allows a partition into plane spanning paths. By Lemma 6.3 the edges of maximal span are partitioned into n wedges and one single edge e_0, with the apexes of the wedges all lying on the same side of the supporting line ℓ of e_0. Assume without loss of generality that $e_0 = (v_0, v_{n+1})$ and that v_i, $1 \leq i \leq n$, is the apex of the wedge of color $c[i]$.

We claim that v_{int}, the unique vertex in $\mathcal{W}$ that is not an extreme point, has to lie in T_a or T_b. Indeed, by the Wedge Lemma, v_{int} cannot lie in any monochromatic wedge. By the Zig-zag Lemma, every point in $Conv(\mathcal{W})$ that is not in T_a or T_b is covered by a monochromatic zig-zag path and therefore lies inside a monochromatic wedge. Thus v_{int} has to lie in T_a or T_b.

So, assume without loss of generality that v_{int} lies in T_a. Move v_{int} to the other side of the edge (v_0, v_1) to get a new point set $\mathcal{V}$ in convex position. As all crossings in $K(\mathcal{W})$ also occur in $K(\mathcal{V})$, $\mathcal{V}$ crossing-dominates $\mathcal{W}$, which concludes the proof. $\square$

However, the statement "$K(\mathcal{P})$ allows a partition into plane spanning paths if and only if $\mathcal{P}$ is crossing-dominated by convex position" is not true for general point sets $\mathcal{P}$. Figure 6.4 shows a point set $\mathcal{Q}$ of size 8 that allows a partition of $K(\mathcal{Q})$ into plane spanning paths. If $\mathcal{Q}$ was crossing-dominated by convex position, then there would be a Hamiltonian cycle in $K(\mathcal{Q})$ consisting only of edges that are not crossed by any other edges [17]. But this is not the case, as can be seen by a simple inspection, so $\mathcal{Q}$ is not crossing-dominated by convex position.

6.1 Proof of the Zig-zag Lemma

Recall that $Ext(\mathcal{W}) = \{v_0, v_1, \ldots, v_{2n}\}$ is enumerated in counter-clockwise order, and that we have already colored the edges of maximal span without loss of generality in such a way that the point v_i, $1 \leq i \leq n$, is the apex of a wedge of color $c[i]$, and the edge $e_0 = (v_0, v_{n+1})$ has color $c[0]$. We also assumed without loss of generality that e_0 is vertical with v_0 above v_{n+1}.

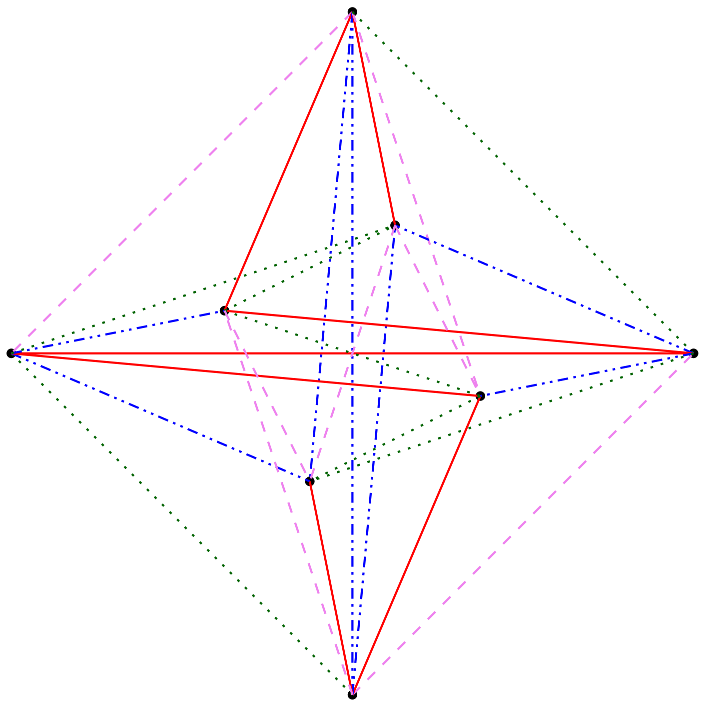


Figure 6.4: A point set that is not crossing-dominated by convex position, but whose complete geometric graph can still be partitioned into plane spanning paths

We want to show that in any partition of $K(\mathcal{W})$ into plane spanning paths, there is a monochromatic zig-zag path for every central color such that these zig-zag paths cover all of $Conv(\mathcal{W})$ except T_a and T_b. Recall that the central colors are $c[\frac{n}{2}]$ and $c[\frac{n}{2}+1]$ if n is even and $c[\frac{n-1}{2}]$, $c[\frac{n+1}{2}]$ and $c[\frac{n+1}{2}+1]$ if n is odd. The triangles T_a and T_b are defined by $\{v_0, v_1, a\}$ and $\{v_n, v_{n+1}, b\}$, respectively, where a is the intersection of (v_0, v_2) and (v_1, v_{2n}), and is b be the intersection of (v_{n-1}, v_{n+1}) and (v_n, v_{n+2}).

We will prove the lemma by successively coloring edges that can only be colored with one color, that is if we were to color the edge with a different color, we would either get a monochromatic crossing, a vertex of degree 3 in some color class, or a contradiction to the Wedge Lemma.

To illustrate the idea of the proof, we first prove the lemma for $n = 5$, i.e. $|Ext(\mathcal{W})| = 11$.

6.1.1 The case $n = 5$

For an illustration of the proof, see Figure 6.5

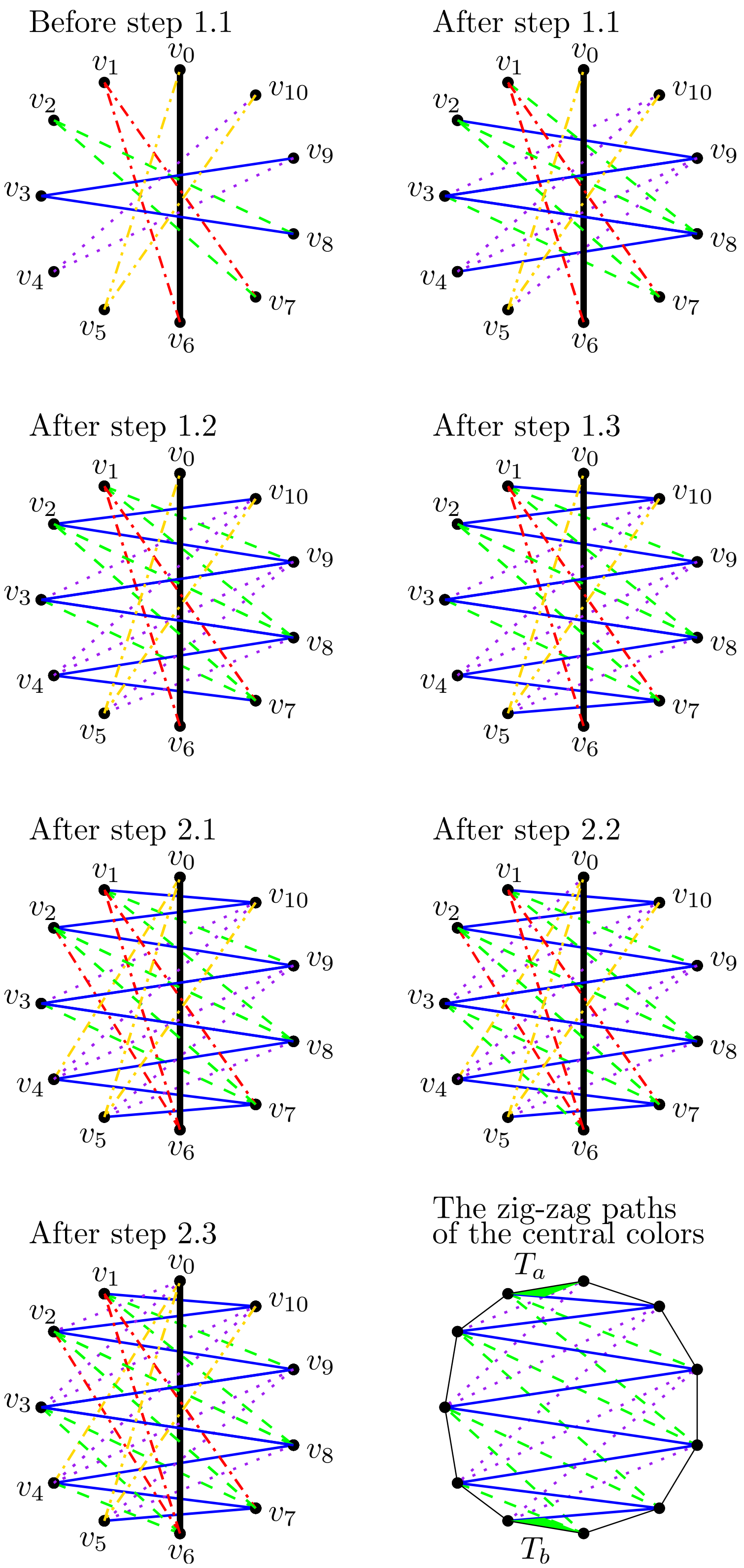


Figure 6.5: An illustration of the proof of the Zig-zag Lemma for $n = 5$. Red edges are drawn dash-dotted, green edges are drawn dashed, blue edges are drawn solid and thin, purple edges are drawn dotted, yellow edges are drawn dash-dot-dotted and black edges are drawn solid and thick.

We have $Ext(\mathcal{W}) = \{v_0, v_1, \ldots, v_{10}\}$, enumerated in counter-clockwise order. Let v_1 be the apex of a red wedge, v_2 be the apex of a green wedge, v_3 be the apex of a blue wedge, v_4 be the apex of a purple wedge, and let v_5 be the apex of a yellow wedge. Finally, let the edge $e_0 = (v_0, v_6)$ be black. As 5 is odd there are three central colors, namely green, blue, and purple.

We will color more edges in two steps, each consisting of 3 smaller steps. In step 1.1 we color all edges of span $n - 1 = 4$ that cross the edge e_0. Similarly, in step 1.2 and 1.3, we color all edges of span 3 and 2, respectively, that cross e_0. Next, in step 2.1, we color the edges of span 4 that are incident to v_0 or v_6 and some other point to the left of e_0. Finally, in steps 2.2 and 2.3, we color the edges of span 3 and 2, respectively, that are incident to v_0 or v_6 and some other point to the left of e_0. We will see that in every step the considered edges can indeed only be colored with one color.

Step 1.1: We want to color all edges of span 4 that cross the edge e_0. We start with the edge (v_3, v_7). This edge crosses the purple edge (v_4, v_9), the yellow edge (v_5, v_{10}), the red edge (v_1, v_6), as well as the black edge e_0. Also, the point v_3 is already incident to two blue edges, thus the edge (v_3, v_7) has to be green. Similarly, we see that (v_4, v_8) has to be blue, (v_5, v_9) has to be purple, (v_3, v_{10}) has to be purple, (v_2, v_9) has to be blue, and (v_1, v_8) has to be green.

Step 1.2: We want to color all edges of span 3 that cross the edge e_0. We start with the edge (v_4, v_7). This edge crosses, amongst others, the yellow edge (v_5, v_{10}), the red edge (v_1, v_6), and the black edge e_0. Also, the point v_4 is already incident to two purple edges and the point v_7 is already incident to two green edges. Thus the edge (v_4, v_7) has to be blue. Similarly, we see that (v_5, v_8) has to be purple, (v_2, v_{10}) has to be blue and (v_1, v_9) has to be green.

Step 1.3: We want to color all edges of span 2 that cross the edge e_0. We start with the edge (v_5, v_7). This edge crosses the red edge (v_1, v_6), and the black edge e_0. Also, the point v_5 is already incident to two yellow edges and two purple edges, and the point v_7 is already incident to two green edges. Thus the edge (v_5, v_7) has to be blue. Similarly, we see that the edge (v_1, v_{10}) also has to be blue.

Step 2.1: We color the edges of span 4 that are incident to v_0 or v_6 and some other point to the left of e_0. We start with the edge (v_2, v_6). This edge crosses, amongst others, the blue edge (v_3, v_8), the purple edge (v_4, v_9), and the yellow edge (v_5, v_{10}). Also, the point v_2 is already incident to two green edges. Thus the edge (v_2, v_6) has to be either red or black. However, if it were black, then the point v_1 would lie inside a black wedge, which is a contradiction to the Wedge Lemma. Thus the edge (v_2, v_6) has to be red. Similarly, we see that the edge (v_0, v_4) has to be yellow.

Step 2.2: We color the edges of span 3 that are incident to v_0 or v_6 and some other point to the left of e_0. We start with the edge (v_3, v_6). This edge crosses, amongst others, the purple edge (v_4, v_9), and the yellow edge (v_5, v_{10}). Also, the point v_3 is already incident to two blue edges and the point v_6 is already incident to two red edges. Again by the Wedge Lemma, the edge (v_3, v_6) cannot be black. Thus (v_3, v_6) has to be green. Similarly, we see that (v_0, v_3) has to be purple.

Step 2.3: We color the edges of span 2 that are incident to v_0 or v_6 and some other point to the left of e_0. We start with the edge (v_4, v_6). This edge crosses, amongst others, the yellow edge (v_5, v_{10}). Also, the point v_4 is already incident to two purple edges and two blue edges and the point v_6 is already incident to two red edges. Again by the Wedge Lemma, the edge (v_4, v_6) cannot be black. Thus (v_4, v_6) has to be green. Similarly, we see that (v_0, v_2) has to be purple.

We see that there is a blue zig-zag path $v_1, v_{10}, v_2, v_9, v_3, v_8, v_4, v_7, v_5$, that covers all of $Conv(\mathcal{W})$ except the two triangles defined by $\{v_0, v_1, v_{10}\}$ and $\{v_5, v_6, v_7\}$, respectively.

Also, there is a purple zig-zag path $v_2, v_0, v_3, v_{10}, v_4, v_9, v_5, v_8$, that covers all of $Conv(\mathcal{W})$ except the triangle defined by $\{v_0, v_1, v_2\}$ and the quadrilateral defined by $\{v_5, v_6, v_7, v_8\}$.

Finally, there is a green zig-zag path $v_9, v_1, v_8, v_2, v_7, v_3, v_6, v_4$, that covers all of $Conv(\mathcal{W})$ except the triangle defined by $\{v_4, v_5, v_6\}$ and the quadrilateral defined by $\{v_0, v_1, v_9, v_{10}\}$.

The two triangles that are not covered by any of these zig-zag paths are exactly the triangles T_a and T_b.

6.1.2 The general case

The proof of the general case is analogous to the proof for $n = 5$, just with more steps. Recall that $Ext(\mathcal{W}) = \{v_0, v_1, \ldots, v_n\}$ is enumerated in counter-clockwise order and that without loss of generality, each v_i, $1 \leq i \leq n$, is the apex of a main wedge of color $c[i]$. Also, we assumed that the edge $e_0 = (v_0, v_{n+1})$ is vertical, with v_0 above v_{n+1}, and has color $c[0]$. Note that at the moment each color class is a zig-zag path of length 1 or 2.

We will again color more edges in two main steps, each now consisting of $n-2$ smaller steps. In the steps $1.k$, $1 \leq k \leq n-2$, we color all edges of span $n-k$ that cross the edge e_0. These edges are the edges (v_i, v_{n+i-k}), for $2+k \leq i \leq n$, and the edges $(v_i, v_{n+i+k+1})$, for $1 \leq i \leq n-k-1$.

Then, in the steps $2.k$, $1 \leq k \leq n-2$, we color the edges of span $n-k$ that are incident to v_0 or v_{n+1} and some other point to the left of e_0. These edges are the edges (v_0, v_{n-k}) and (v_{k+1}, v_{n+1}).

Step 1: We will distinguish whether k is even or odd. We will maintain the following invariants:

After step 1.k, where k is odd, we have that

(o1) for $2+k \leq i \leq n$, the point v_i is incident, amongst others, to two edges of each color $c[i]$, $c[i-1]$, …, $c[i-\frac{k-1}{2}]$ and one edge of color $c[i-\frac{k+1}{2}]$,

(o2) for $2 \leq j \leq n-k$, the point v_{n+j} is incident, amongst others, to two edges of each color $c[j]$, $c[j+1]$, …, $c[j+\frac{k-1}{2}]$,

(o3) for $1 \leq i \leq n-k-1$, the point v_i is incident, amongst others, to two edges of each color $c[i]$, $c[i+1]$, …, $c[i+\frac{k-1}{2}]$ and one edge of color $c[i+\frac{k+1}{2}]$, and

(o4) for $2+k \leq j \leq n$, the point v_{n+j} is incident, amongst others, to two edges of each color $c[j-1]$, $c[j-2]$, …, $c[j-\frac{k+1}{2}]$.

After step 1.k, where k is even, we have that

(e1) for $2+k \leq i \leq n$, the point v_i is incident, amongst others, to two edges of each color $c[i]$, $c[i-1]$, …, $c[i-\frac{k}{2}]$,

(e2) for $2 \leq j \leq n-k$, the point v_{n+j} is incident, amongst others, to two edges of each color $c[j]$, $c[j+1]$, …, $c[j+\frac{k}{2}-1]$ and one edge of color $c[j+\frac{k}{2}]$,

(e3) for $1 \leq i \leq n-k-1$, the point v_i is incident, amongst others, to two edges of each color $c[i]$, $c[i+1]$, …, $c[i+\frac{k}{2}]$, and

(e4) for $2+k \leq j \leq n$, the point v_{n+j} is incident, amongst others, to two edges of each color $c[j-1]$, $c[j-2]$, …, $c[j-\frac{k}{2}]$ and one edge of color $c[j-\frac{k}{2}-1]$.

Of course there are points that satisfy for example both (e1) and (e3) after some step. The reason we state the invariants like this is that in this form each invariant contains exactly the information that we will need.

Note that before step 1.1, the invariants (e1)-(e4) hold for $k=0$: each point v_i, $1 \leq i \leq n$, is incident to two edges of color $c[i]$, which implies (e1) and (e3), and each point v_{n+i}, $2 \leq i \leq n$, is incident to one edge of color $c[i]$ and one edge of color $c[i-1]$, which implies (e2) and (e4), respectively, as the sequences of colors for two edges in this two invariants are empty for $k=0$.

We will also maintain that every color class is a zig-zag path.

Step 1.k, k odd: We want to color all edges of span $n-k$ that cross the edge e_0. We start with the edges (v_i, v_{n+i-k}), for $2+k \leq i \leq n$. For any m with $i+1 \leq m \leq n$, the edge (v_i, v_{n+i-k}) crosses the edge (v_m, v_{n+m+1}), which has color $c[m]$. Also, for any l with $1 \leq l \leq i-k-1$, the edge (v_i, v_{n+i-k})

crosses the edge (v_l, v_{n+l}), which has color $c[l]$. By invariant (e1), the vertex v_i is incident to two edges of each color $c[i]$, $c[i-1]$, ..., $c[i-\frac{k-1}{2}]$. By setting $j = i-k$ in invariant (e2), we deduce that the vertex v_{n+i-k} is incident to two edges of each color $c[i-k]$, $c[i-k+1]$, ..., $c[i-k+\frac{k-1}{2}-1]$ and one edge of color $c[i-k+\frac{k-1}{2}] = c[i-\frac{k+1}{2}]$. As the edge (v_i, v_{n+i-k}) also crosses e_0, which has color $c[0]$, (v_i, v_{n+i-k}) must have color $c[i-\frac{k+1}{2}]$ and the invariant (o1) holds. Substituting $i = j+k$, we deduce that invariant (o2) holds as well.

Note that if n is odd, we color the edge (v_n, v_{n+2}) with color $c[\frac{n+1}{2}]$ in step 1.$(n-2)$.

Now, consider the edge $(v_i, v_{n+i+k+1})$, for $1 \leq i \leq n-k-1$. For any m with $i+k+1 \leq m \leq n$, the edge $(v_i, v_{n+i+k+1})$ crosses the edge (v_m, v_{n+m+1}), which has color $c[m]$. Also, for any l with $1 \leq l \leq i-1$, the edge $(v_i, v_{n+i+k+1})$ crosses the edge (v_l, v_{n+l}), which has color $c[l]$. By invariant (e3), the vertex v_i is incident to two edges of each color $c[i]$, $c[i+1]$, ..., $c[i+\frac{k-1}{2}]$. By setting $j = i+k+1$ in invariant (e4), we deduce that the vertex $v_{n+i+k+1}$ is incident to two edges of each color $c[i+k]$, $c[i+k-1]$, ..., $c[i+k+1-\frac{k-1}{2}]$ and one edge of color $c[i+k+1-\frac{k-1}{2}-1] = c[i+\frac{k+1}{2}]$. As the edge $(v_i, v_{n+i+k+1})$ also crosses e_0, which has color $c[0]$, $(v_i, v_{n+i+k+1})$ must have color $c[i+\frac{k+1}{2}]$ and the invariant (o3) holds. Substituting $i = j-k-1$, we deduce that invariant (o4) holds as well.

Note that if n is odd, we color the edge (v_1, v_{2n}) with color $c[\frac{n+1}{2}]$ in step 1.$(n-2)$.

None of the edges that we have colored is disjoint from its color class and we have not colored any edges of span 1. Also, every newly colored edge (v_i, v_{n+i-k}) is incident to exactly one edge (v_{i-1}, v_{n+i-k}) of the same color, and every newly colored edge $(v_i, v_{n+i+k+1})$ is incident to exactly one edge $(v_{i-1}, v_{n+i+k+1})$ of the same color. Thus each color class is still a zig-zag path. This concludes Step 1.k for k odd.

Step 1.k, k even: We want to color all edges of span $n-k$ that cross the edge e_0. We start with the edge (v_i, v_{n+i-k}), for $2+k \leq i \leq n$. For any m with $i+1 \leq m \leq n$, the edge (v_i, v_{n+i-k}) crosses the edge (v_m, v_{n+m+1}), which has color $c[m]$. Also, for any l with $1 \leq l \leq i-k-1$, the edge (v_i, v_{n+i-k}) crosses the edge (v_l, v_{n+l}), which has color $c[l]$. By setting $j = i-k$ in invariant (o2), we deduce that the vertex v_{n+i-k} is incident to two edges of each color $c[i-k]$, $c[i-k+1]$, ..., $c[i-k+\frac{k-2}{2}] = c[i-\frac{k}{2}-1]$. By invariant (o1) the vertex v_i is incident to two edges of each color $c[i]$, $c[i-1]$, ..., $c[i-\frac{k-2}{2}]$ and one edge of color $c[i-\frac{k}{2}]$. As the edge (v_i, v_{n+i-k}) also crosses e_0, which has color $c[0]$, (v_i, v_{n+i-k}) must have color $c[i-\frac{k}{2}]$ and the invariant (e1) holds. Substituting $i = j+k$, we deduce that invariant (e2) holds as well.

Note that if n is even, we color the edge (v_n, v_{n+2}) with color $c[\frac{n}{2}+1]$ in step 1.$(n-2)$.

Now, consider the edge $(v_i, v_{n+i+k+1})$, for $1 \le i \le n-k-1$. For any m with $i+k+1 \le m \le n$, the edge $(v_i, v_{n+i+k+1})$ crosses the edge (v_m, v_{n+m+1}), which has color $c[m]$. Also, for any l with $1 \le l \le i-1$, the edge $(v_i, v_{n+i+k+1})$ crosses the edge (v_l, v_{n+l}), which has color $c[l]$. By setting $j = i+k+1$ in invariant (o4), we deduce that the vertex $v_{n+i+k+1}$ is incident to two edges of each color $c[i+k]$, $c[i+k-1]$, …, $c[i+k+1-\frac{k}{2}] = c[i+\frac{k}{2}+1]$. By invariant (o3) the vertex v_i is incident to two edges of each color $c[i]$, $c[i+1]$, …, $c[i+\frac{k-2}{2}]$ and one edge of color $c[i+\frac{k}{2}]$. As the edge $(v_i, v_{n+i+k+1})$ also crosses e_0, which has color $c[0]$, $(v_i, v_{n+i+k+1})$ must have color $c[i+\frac{k}{2}]$ and the invariant (e3) holds. Substituting $i = j-k-1$, we deduce that invariant (e4) holds as well.

Note that if n is even, we color the edge (v_1, v_{2n}) with color $c[\frac{n}{2}]$ in step 1.$(n-2)$.

None of the edges that we have colored is disjoint from its color class and we have not colored any edges of span 1. Also, every newly colored edge (v_i, v_{n+i-k}) is incident to exactly one edge $(v_i, v_{n+i-k+1})$ of the same color, and every newly colored edge $(v_i, v_{n+i+k+1})$ is incident to exactly one edge (v_i, v_{n+i+k}) of the same color. Thus each color class is still a zig-zag path. This concludes Step 1.k for k even.

Step 2: We continue with the steps 2.k, $1 \le k \le n-2$. Recall that in step 2.k we want to color the edges of span $n-k$ that are incident to v_0 or v_{n+1} and some other point to the left of e_0, i.e. the edges (v_0, v_{n-k}) and (v_{k+1}, v_{n+1}).

In steps 2.k, we will maintain the following invariants:

After step 2.k, where k is odd, we have that

(o5) the vertex v_0 is incident to two edges of each color $c[n]$, $c[n-1]$, …, $c[n-\frac{k-1}{2}]$, and the vertex v_{n+1} is incident to two edges of each color $c[1]$, $c[2]$, …, $c[\frac{k+1}{2}]$.

After step 2.k, where k is even, we have that

(e5) the vertex v_0 is incident to two edges of each color $c[n]$, $c[n-1]$, …, $c[n-\frac{k}{2}+1]$ and one edge of color $c[n-\frac{k}{2}]$, and the vertex v_{n+1} is incident to two edges of each color $c[1]$, $c[2]$, …, $c[\frac{k}{2}]$ and one edge of color $c[\frac{k}{2}+1]$.

Of course, the invariants (o1)-(o4) and (e1)-(e4) still hold, but note that for any v_i, the invariants (e1) and (o1) are defined after step 1.k only for $2+k \le i \le n$, and similarly for the other invariants. Again, before step 2.1 the invariant (e5) holds for $k=0$, with the sequences of colors for two edges being empty.

We will again maintain that every color class is a zig-zag path.

Step 2.k, k odd: We want to color the edges (v_0, v_{n-k}) and (v_{k+1}, v_{n+1}). We start with the edge (v_0, v_{n-k}). We first note that (v_0, v_{n-k}) cannot have color $c[0]$, as else v_n would lie inside a monochromatic wedge. For any j with $1 \leq j \leq n-k-1$, the edge (v_0, v_{n-k}) crosses the edge (v_j, v_{n+j}), which has color $c[j]$. By invariant (e3), which is defined up to step 1.$(k-1)$ for v_{n-k}, the point v_{n-k} is incident to two edges of each color $c[n-k]$, $c[n-k+1]$, …, $c[n-k+\frac{k-1}{2}] = c[n-\frac{k+1}{2}]$. By invariant (e5), the point v_0 is incident to two edges of each color $c[n]$, $c[n-1]$, …, $c[n-\frac{k-1}{2}+1]$ and one edge of color $c[n-\frac{k-1}{2}]$. Thus the edge (v_0, v_{n-k}) must have color $c[n-\frac{k-1}{2}]$ and the first half of invariant (o5) holds.

Note that if n is odd, we color the edge (v_0, v_2) with color $c[\frac{n+1}{2}+1]$ in step 1.$(n-2)$.

Now we color the edge (v_{k+1}, v_{n+1}). We again note that (v_{k+1}, v_{n+1}) cannot have color $c[0]$, as else v_1 would lie inside a monochromatic wedge. For any j with $k+2 \leq j \leq n$, the edge (v_{k+1}, v_{n+1}) crosses the edge (v_j, v_{n+j}), which has color $c[j]$. By invariant (e1), which is defined up to step 1.$(k-1)$ for v_{k+1}, the point v_{k+1} is incident to two edges of each color $c[k+1]$, $c[k]$,…, $c[k+1-\frac{k-1}{2}] = c[\frac{k+1}{2}+1]$. By invariant (e5), the point v_{n+1} is incident to two edges of each color $c[1]$, $c[2]$, …, $c[\frac{k-1}{2}]$ and one edge of color $c[\frac{k-1}{2}+1] = c[\frac{k+1}{2}]$. Thus the edge (v_{k+1}, v_{n+1}) must have color $c[\frac{k+1}{2}]$ and the second half of invariant (o5) also holds.

Note that if n is odd, we color the edge (v_{n-1}, v_{n+1}) with color $c[\frac{n-1}{2}]$ in step 1.$(n-2)$.

None of the edges that we have colored is disjoint from its color class and we have not colored any edges of span 1. Also, every newly colored edge (v_0, v_{n-k}) is incident to exactly one edge (v_0, v_{n-k+1}) of the same color, and every newly colored edge (v_{k+1}, v_{n+1}) is incident to exactly one edge (v_k, v_{n+1}) of the same color. Thus each color class is still a zig-zag path. This concludes Step 2.k for k odd.

Step 2.k, k even: We again want to color the edges (v_0, v_{n-k}) and (v_{k+1}, v_{n+1}). We start with the edge (v_0, v_{n-k}). We first note that (v_0, v_{n-k}) cannot have color $c[0]$, as else v_n would lie inside a monochromatic wedge. For any j with $1 \leq j \leq n-k-1$, the edge (v_0, v_{n-k}) crosses the edge (v_j, v_{n+j}), which has color $c[j]$. By invariant (o3), which is defined up to step 1.$(k-1)$ for v_{n-k}, the point v_{n-k} is incident to two edges of each color $c[n-k]$, $c[n-k+1]$, …, $c[n-k+\frac{k-2}{2}] = c[n-\frac{k}{2}-1]$ and one edge of color $c[n-\frac{k}{2}]$. By invariant (o5), the point v_0 is incident to two edges of each color $c[n]$, $c[n-1]$, …, $c[n-\frac{k-2}{2}] = c[n-\frac{k}{2}+1]$. Thus the edge (v_0, v_{n-k}) must have color $c[n-\frac{k}{2}]$ and the first half of invariant (e5) holds.

Note that if n is even, we color the edge (v_0, v_2) with color $c[\frac{n}{2}+1]$ in step 1.$(n-2)$.

Now we color the edge (v_{k+1}, v_{n+1}). We again note that (v_{k+1}, v_{n+1}) cannot have color $c[0]$, as else v_1 would lie inside a monochromatic wedge. For any j with $k+2 \leq j \leq n$, the edge (v_{k+1}, v_{n+1}) crosses the edge (v_j, v_{n+j}), which has color $c[j]$. By invariant (o1), which is defined up to step 1.$(k-1)$ for v_{k+1}, the point v_{k+1} is incident to two edges of each color $c[k+1]$, $c[k]$, ..., $c[k+1-\frac{k-2}{2}] = c[\frac{k}{2}+2]$ and one edge of color $c[\frac{k}{2}+1]$. By invariant (o5), the point v_{n+1} is incident to two edges of each color $c[1]$, $c[2]$, ..., $c[\frac{k}{2}]$. Thus the edge (v_{k+1}, v_{n+1}) must have color $c[\frac{k}{2}+1]$ and the second half of invariant (e5) also holds.

Note that if n is even, we color the edge (v_{n-1}, v_{n+1}) with color $c[\frac{n}{2}]$ in step 1.$(n-2)$.

None of the edges that we have colored is disjoint from its color class and we have not colored any edges of span 1. Also, every newly colored edge (v_0, v_{n-k}) is incident to exactly one edge (v_{2n}, v_{n-k}) of the same color, and every newly colored edge (v_{k+1}, v_{n+1}) is incident to exactly one edge (v_{k+1}, v_{n+2}) of the same color. Thus each color class is still a zig-zag path. This concludes Step 2.k for k even.

Wrapping up: We have now finished coloring the edges that we wanted to color. It remains to show that the zig-zag paths of the central colors cover all of $Conv(\mathcal{W})$ except the two triangles T_a and T_b. We distinguish whether n is even or odd.

We start with the case that n is odd, i.e. the central colors are $c[\frac{n-1}{2}]$, $c[\frac{n+1}{2}]$ and $c[\frac{n+1}{2}+1]$. Recall that in step 1.$(n-2)$ we have colored the edges (v_1, v_{2n}) and (v_n, v_{n+2}) with color $c[\frac{n+1}{2}]$. As each color class is a zig-zag path, this implies that the zig-zag path of color $c[\frac{n+1}{2}]$ already covers all of $Conv(\mathcal{W})$ except the two triangles T_1 and T_2, defined by $\{v_0, v_1, v_{2n}\}$ and $\{v_n, v_{n+1}, v_{n+2}\}$, respectively. Recall that in step 2.$(n-2)$ we have colored the edge (v_0, v_2) with color $c[\frac{n+1}{2}+1]$ and the edge (v_{n-1}, v_{n+1}) with color $c[\frac{n-1}{2}]$. As each color class is a zig-zag path, this implies that the zig-zag path of color $c[\frac{n+1}{2}+1]$ covers all of T_1 except T_a. Analogously, the zig-zag path of color $c[\frac{n-1}{2}]$ covers all of T_2 except T_b. Thus the zig-zag paths of the central colors cover all of $Conv(\mathcal{W})$ except the two triangles T_a and T_b, which is what we claimed.

Finally, we consider the case where n is even, i.e. the central colors are $c[\frac{n}{2}]$ and $c[\frac{n}{2}+1]$. Recall that in step 1.$(n-2)$ we have colored the edge (v_1, v_{2n}) with color $c[\frac{n}{2}]$ and (v_n, v_{n+2}) with color $c[\frac{n}{2}+1]$. Recall that in step 2.$(n-2)$ we have colored the edge (v_0, v_2) with color $c[\frac{n}{2}+1]$ and (v_{n-1}, v_{n+1}) with color $c[\frac{n}{2}]$. As each color class is a zig-zag path, this implies that the zig-zag path of color $c[\frac{n}{2}]$ covers all of $Conv(\mathcal{W})$ except the two triangles T_1 and T_2, defined by $\{v_0, v_1, v_{2n}\}$ and $\{v_{n-1}, v_n, v_{n+1}\}$, respectively. Analogously, the zig-zag path of color $c[\frac{n}{2}+1]$ covers all of $Conv(\mathcal{W})$ except the two triangles T_3 and T_4, defined by $\{v_0, v_1, v_2\}$ and $\{v_n, v_{n+1}, v_{n+2}\}$, respectively. Thus, the two zig-zag path of the central colors cover all of $Conv(\mathcal{W})$ except $T_1 \cap T_3 = T_a$ and $T_2 \cap T_4 = T_b$. This concludes the proof.

Chapter 7

Coloring Line Segment Arrangements

In this chapter we consider the more general problem of coloring the segments in a line segment arrangement in a way such that no two crossing segments get the same color. Of course, every line segment arrangement can be interpreted as a geometric graph, the vertices being the endpoints of the segments.

More specifically, we consider the following problem: Given an arrangement of line segments, where some segments are already colored, can we extend this partial coloring to a coloring of the whole arrangement in a way that we do not get any monochromatic crossings? We first only allow three colors, red, blue and green, but we later generalize the results for more colors.

Theorem 7.1 *It is $\mathcal{NP}$-complete to decide whether a partial 3-coloring of a line segment arrangement can be extended to a complete 3-coloring of the arrangement without monochromatic crossings.*

Proof We will first prove $\mathcal{NP}$-hardness by reduction from Planar 3-SAT [15]. For any Planar 3-SAT formula we construct a partially 3-colored line segment arrangement with the property that the partial 3-coloring can be extended to a complete 3-coloring without monochromatic crossings if and only if the Planar 3-SAT formula is satisfiable.

Let F be a Planar 3-SAT formula and let $G(F)$ be its associated graph, whose vertex set consists of a vertex v_X for every variable X and a vertex v_C for every clause C, with an edge between v_X and v_C if and only if X or $\neg X$ appears in C. By definition of Planar 3-SAT, $G(F)$ is planar. Consider a plane straight-line drawing of $G(F)$. Note that Fáry's theorem assures that such a drawing always exists. We will mimic the formula F by constructing partially 3-colored line segment-configurations, called *gadgets*, that serve as variables, negations and disjunctions, and concatenating them according to the graph $G(F)$.

We construct a value gadget as an uncolored segment, crossed by a short green segment. In the whole proof, we call a segment *short* if it only crosses one segment of the arrangement. The uncolored segment can thus only be colored red or blue, and we will interpret red as ”1” and blue as ”0”. For the variable gadgets, we can thus use value gadgets. For a NOT-gadget, we just cross two value gadgets. Similarly, we can also construct turns or copy a variable, if needed. See Figure 7.1 for a drawing of these constructions.

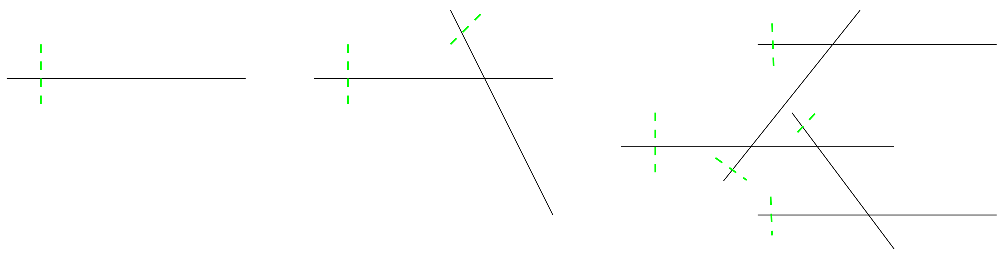


Figure 7.1: A value-gadget (left), a NOT-gadget (middle) and a gadget to copy a variable (right). Green segments are drawn dashed.

We now construct an OR-gadget, as depicted in Figure 7.2. Let X and Y be the input values and let Z be the output value. Denote the uncolored segments corresponding to the values X, Y and Z by x, y and z, respectively. Place x, y and z such that z lies between x and y. Then, place two uncolored segments a and b such that a crosses x and z, b crosses y and z and a and b cross each other. Next, place four uncolored segments c, d, e and f, such that c crosses x and z, d crosses x and c, e crosses y and z, and f crosses y and e. Finally, cross c and e with short red segments, as well as d and f with short blue segments.

We will now show that this construction indeed is an OR-gadget. First assume that both input value segments x and y have the same color, without loss of generality red. Then both a and b can only be blue or green, and as they cross, one must be blue. Thus the output value segment z must be red again. We can complete the coloring by coloring d and f green and c and e blue. Now, assume that the input value segments have different colors, without loss of generality say that x is red and y is blue. Then d must be green and therefore c has to be blue, implying that z is again red. Also, e and f must be green and red, respectively. We can finish the coloring by coloring b green and a blue. Summarizing, if both input value segments are red, or one of them is red and the other one blue, then the output value segment must be red. If however both input value segments are blue, the also the output value segment has to be blue. Thus our construction is indeed an OR-gadget.

To build a clause gadget we can concatenate two OR-gadgets, taking the

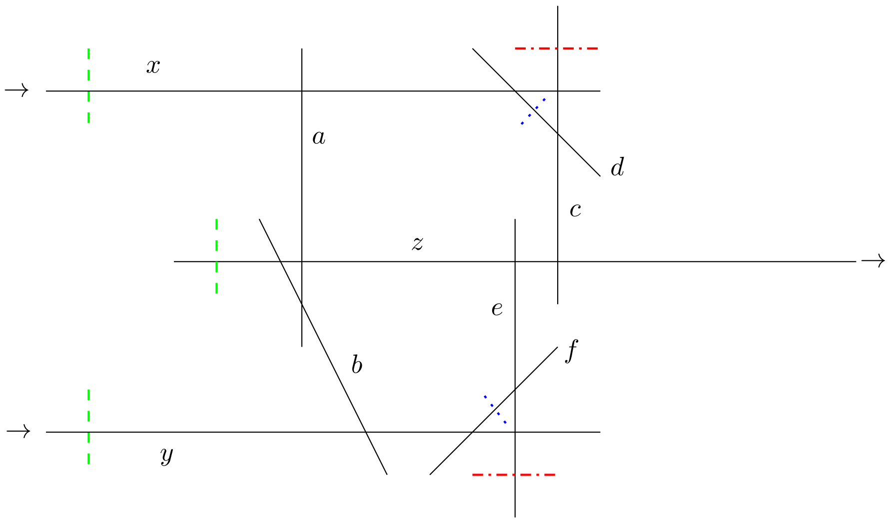


Figure 7.2: An OR-gadget. Green segments are drawn dashed, blue segments are dotted, and red segments are dash-dotted.

output value segment of the first one as input value segment of the second one, and then cross the last output value segment with a short blue segment, enforcing that the clause must be satisfied. Thus, if we can extend the partial coloring to a complete coloring without monochromatic crossings, then the Planar 3-SAT formula is satisfiable and a satisfiable assignment is given by the colors of the variable segments. On the other hand, given a satisfying assignment, coloring the value segments accordingly induces a larger partial coloring of the line segment arrangement, where the only uncolored edges are in the OR-gadgets. As shown above, the coloring can then be completed without getting any monochromatic crossings. As each gadget only requires a constant number of segments and each planar graph can be drawn with straight lines on a grid of polynomial size in polynomial time, the line segment arrangement can be constructed in polynomial time, which finishes the $\mathcal{NP}$-hardness proof.

On the other hand, given a complete coloring of a line segment arrangement, we can check whether there are monochromatic crossings in polynomial time, as there are only polynomially many crossings. Thus the problem is in $\mathcal{NP}$, which completes the proof. $\square$

As every line segment arrangement also is a geometric graph, we immediately get the following corollary:

Corollary 7.2 *It is $\mathcal{NP}$-complete to decide whether a partial 3-coloring of the edges of a geometric graph can be extended to a complete 3-coloring without monochromatic crossings.*

We can also do the same for more than three colors:

Theorem 7.3 *For any $k \in \mathbb{N}$, $k \geq 3$, it is $\mathcal{NP}$-complete to decide whether a partial k-coloring of a line segment arrangement can be extended to a complete k-coloring of the arrangement without monochromatic crossings.*

Proof Do the same construction as in the proof of Theorem 7.1, with the slight modification that all uncolored segments are also crossed by a short segment of every color except red, blue and green. As k is a fixed constant, all gadgets still have constant size. Again, every uncolored segment can only be colored red, blue or green, just as in the proof of Theorem 7.1, and all arguments work analogously. □

We can again state this result in terms of geometric graphs.

Corollary 7.4 *For any $k \in \mathbb{N}$, $k \geq 3$, it is $\mathcal{NP}$-complete to decide whether a partial k-coloring of the edges of a geometric graph can be extended to a complete k-coloring without monochromatic crossings.*

Chapter 8

Conclusion

We have reduced the problem of finding a partition of a complete geometric graph into plane spanning double stars to the problem of finding an expandable perfect matching on its vertex set. We gave a necessary, as well as a sufficient condition for a matching to be expandable and we proved that it can be decided in polynomial time whether a matching is expandable. However, the necessary and the sufficient condition are not the same and hence we still lack a nice characterization of expandable matchings.

Question 8.1 *Can we characterize all expandable matchings?*

We also showed that finding a large packing with plane spanning double stars is equivalent to finding a large expandable matching. We used this to construct large packings for complete geometric graphs drawn on a few special point sets.

Question 8.2 *How many plane spanning double stars can be packed into any complete geometric graph?*

One way to give a lower bound for this number would be to find a large set $\mathcal{L}$ of pairwise non-parallel lines with the property that there is at least one vertex of the complete geometric graph in each unbounded region formed by $\mathcal{L}$.

Question 8.3 *Let $\mathcal{P}$ be a point set and let $\mathcal{L}$ be a set of lines such that there is at least one point of $\mathcal{P}$ in every unbounded region formed by $\mathcal{L}$. How large can $\mathcal{L}$ be?*

The author suspects that $\mathcal{L}$ could always have linear size, which would imply that a linear number of plane spanning double stars could be packed into any complete geometric graph.

We also found some point sets whose complete geometric graphs cannot be partitioned into plane spanning double stars. This of course raises the following question:

Question 8.4 *Can the complete geometric graphs that do not allow a partition into plane spanning double stars still be partitioned into plane spanning trees?*

This question can be answered positively for the bumpy wheel BW_{10}, but remains open for larger graphs.

As for partitions into plane spanning paths, we showed that a complete geometric graph, drawn on a point set with exactly one point not on the boundary of the convex hull, can be partitioned into plane spanning paths if and only if the point set is crossing-dominated by convex position. We also gave an example which shows that this is not true for general point sets. Interestingly, the partition into plane spanning paths in this example is combinatorially different from any partition into plane spanning paths of a complete geometric graph drawn on a point set in convex position.

Question 8.5 *Let $\mathcal{P}$ be a point set such that $K(\mathcal{P})$ allows a partition into plane spanning paths that is combinatorially equivalent to a partition into plane spanning paths of a complete geometric graph drawn on a point set in convex position. Does this imply that $\mathcal{P}$ is crossing-dominated by convex position?*

Question 8.6 *Can we characterize the point sets whose complete geometric graphs allow a partition into plane spanning paths?*

Finally, the question that motivated this thesis remains open:

Question 8.7 *Does every complete geometric graph with an even number of vertices allow a partition of its edge set into plane spanning trees?*

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